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Uncertainty principles for hypercomplex signals in the linear canonical transform domains

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ABSTRACT

Linear canonical transforms (LCTs) are a family of integral transforms with wide application in optical, acoustical, electromagnetic, and other wave propagation problems. The Fourier and fractional Fourier transforms are special cases of LCTs. In this paper, we extend the uncertainty principle for hypercomplex signals in the linear canonical transform domains, giving the tighter lower bound on the product of the effective widths of complex paravector- (multivector-)valued signals in the time and frequency domains. It is seen that this lower bound can be achieved by a Gaussian signal. An example is given to verify the result.

1. Introduction

Uncertainty principle in the time–frequency plane plays an important role in signal processing [14,17,22,23,32,38,44,45,47,54,55,60]. This principle states that for a given unit energy signal $f(t)$ with Fourier transform

$$\hat{f}(\omega):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-\mathbf{i}\omega t}\,dt,$$

the product of spreads of the signal in time domain and frequency domain is bounded by a lower bound:

$$\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4}, \tag{1}$$

where σ_t and σ_ω are the duration and bandwidth of a signal $f(t)$ defined by

$$\sigma_t^2:=\int_{-\infty}^{\infty}(t-\langle t\rangle)^2|f(t)|^2\,dt$$

and

$$\sigma_\omega^2:=\int_{-\infty}^{\infty}(\omega-\langle\omega\rangle)^2|\hat{f}(\omega)|^2\,d\omega,$$

respectively. Here

$$\langle t\rangle:=\int_{-\infty}^{\infty} t|f(t)|^2\,dt$$

is the mean time and

$$\langle\omega\rangle:=\int_{-\infty}^{\infty}\omega|\hat{f}(\omega)|^2\,d\omega$$

is the mean frequency. Without loss of generality, let $\langle t\rangle=0$ and $\langle\omega\rangle=0$, then the essence of uncertainty principle (1) will not be affected. The reason is that the standard derivation does not depend on the mean because it is defined as the broadness about the mean [10,18]. Consequently, Eq. (1) becomes

$$\left(\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt\right)\left(\int_{-\infty}^{\infty}\omega^2|\hat{f}(\omega)|^2\,d\omega\right)\geq\frac{1}{4}.$$

For the importance of uncertainty principle in physics [1,10,28–30,39,40,51,56,59], there are many efforts to extend it to various types of functions and integral transformations.

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The linear canonical transform (LCT) being a generalization of the Fourier transform (FT), the fractional Fourier transform (FrFT) and the Fresnel transform [48,44], was first proposed in the 1970s by Moshinsky and Collins [13,43]. It is an effective processing tool for chirp signal analysis, such as the LCT filtering [21,49,58], the parameter estimation and sampling progress for non-bandlimited signals with non-linear Fourier atoms [37]. The windowed LCT [35], with a local window function, can reveal the local LCT-frequency contents, and it enjoys high concentrations and eliminates cross terms. The analogue of the Poisson summation formula, sampling formulas, series expansions and Paley–Wiener theorem is studied in [35,36]. Serving as a powerful analyzing tool, LCT has been widely used in many fields such as optics and signal processing [3,43,50,57]. Recently, researchers discussed the uncertainty relations for FrFT [44,49,54] and LCT [36,55,53,56] and tried to derive sharp uncertainty principle bounds for them. A stronger uncertainty principle in LCT involving the phase derivative of the signal was discussed in [15].

In view of numerous applications, one is particularly interested in higher dimensional analogues to Euclidean space $\mathbf{R}^n$. Higher dimensional extensions of the LCT have been studied, for instance, in [33,34]. The LCT is first extended to the Clifford algebra $Cl_{0,m}$ (see the notation in Section 2) in [34]. It is used to study the generalized prolate spheroidal wave functions and the connection with energy concentration problems [34]. The present work develops the definition of LCT (see Definition 3.1 in Section 3) which uses the imbedding of $\mathbf{R}^m$ into the (complex) Clifford algebra $Cl_{0,m}$. The Clifford algebra $Cl_{0,m}$ provides $\mathbf{R}^m$ with a global complex structure in analogy with the imbedding of $\mathbf{R}$ into the complex plane $\mathbf{C}$. Under this frame we present in this note the precise analogue of the classical Heisenberg uncertainty principles in linear canonical domains which have been targeted by others.

In Hamiltonian quaternion analysis some papers combined the uncertainty relations and the quaternionic Fourier transforms (QFTs) [2,7,25,46]. Due to the non-commutative property of multiplication of quaternions, there are different types of QFTs: double-sided (two-sided) QFT, left-sided QFT and right-sided QFT. The QFT plays a vital role in the representation of hypercomplex signals. It transforms a real (or quaternionic) 2D signal into a quaternion-valued frequency domain signal. The four components of the QFT separate four cases of symmetry into real signals instead of only two as in the complex Fourier transform. In [52] the authors used the QFT to proceed color image analysis. The paper [6] implemented the QFT to design a color image digital watermarking scheme. The QFT are applied to image pre-processing and neural computing techniques for speech recognition [5]. It is well-known that the Plancherel theorem is not valid for the double-sided or the left-sided QFT [2]. For this reason, many studies focus on the right-sided QFT. In [19,20], certain asymptotic properties of the (right-sided) QFT are analyzed and generalizations of classical Bochner–Minlos theorems to the framework of quaternionic analysis are derived. The uncertainty principle for the (right-sided) QLCT, the generalization of the (right-sided) QFT in the Hamiltonian quaternion algebra, is recently derived in [33]. In 2010, Hitzer [25] studied the $\pm$ split of quaternions and its effects on the double-sided QFT. Based on this he formulated the directional uncertainty principle for QFT of quaternion-valued 2D signal. The QFT belongs to the growing family of Clifford Fourier transformations. But the left and right placement of the exponential factors in quaternion Fourier kernel distinguishes it. Therefore the (double-sided) QFT [25] is not the special case of Clifford FT [26].

In Clifford algebra, Hitzer et al. [26,27,41,42] investigated a directional uncertainty principle for the Clifford Fourier transform, which describes how the variances (in arbitrary but fixed directions) of a multivector-valued function and its Clifford Fourier transform are related. In [26,27], the research was studied for the Clifford- ($Cl_{0,m}, m=2,3 \pmod 4$-)valued signals, while the latter [41,42] focused on the octonion-($Cl_{0,3}$-)valued signals. Using the scalar-valued phase derivative of hypercomplex signals [62], two uncertainty principles, of which one is for scalar-valued hypercomplex signals and the other is for axial form hypercomplex signals, for Fourier transforms were studied in [61]. To the best of our knowledge, a systematic work on the investigation of uncertainty relations using the LCT of a paravector- (multivector-)valued function has not been carried out yet.

In the present work, we study the LCT in Euclidean space which transforms a paravector-valued space signal into a complex paravector-valued frequency signal. Some important properties of the LCT are analyzed. Two uncertainty principles for the LCT of complex multivector-valued signals are established. These uncertainty principles prescribe the lower bounds on the products of the effective widths of complex multivector-valued signals in the space and frequency domains. The main motivation of the present study is to develop further general methods for time–frequency analysis, developing sampling theory in the m-D case, filter design, signal synthesis and optics in the Clifford algebra. We note that the present theory for the paravector- (multivector-)valued signals can also be generalized to the complex Clifford- ($Cl_{0,m}$-)valued signals without difficulties.

The article is organized as follows. In Section 2 we provide the basic knowledge of Clifford algebra used in the paper. Then the LCT of complex paravector-valued signal is introduced and studied in Section 3. Some important properties such as Parseval theorem are obtained. They are necessary to prove the uncertainty principle in the LCT domain. In Section 4 we formulate and prove the classical Heisenberg uncertainty principles for the LCT of complex paravector-valued signal. These principles prescribe lower bounds on the products of the effective widths of paravector-valued signals in the time and frequency domains. We give an example to verity the result in Section 5. Some conclusions are drawn in Section 6.

2. Clifford algebra

The theory of Clifford algebras is intimately connected with the theory of quadratic forms and orthogonal transformations. They generalize the real numbers, complex numbers, quaternions and several other hypercomplex number systems [11,12]. Clifford algebras have important applications in a variety of fields including geometry, theoretical physics and digital image processing. They are named after the English geometer William Kingdon Clifford [11,12]. In the present section, we begin by reviewing some definitions and basic properties of Clifford algebra [9,16].

For all what follows we will work in $\mathbf{R}^m$ ($\mathbf{C}^m$), the usual Euclidean space (several complex variables space). This means we can express each element $\underline{x} \in \mathbf{R}^m(\mathbf{C}^m)$ uniquely in the form:

$$\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m, \quad x_j \in \mathbf{R}(\mathbf{C}) \quad (j = 1, 2, \cdots, m)$$

where the units $\mathbf{e}_1, \ldots, \mathbf{e}_m$ are *basic elements* of $\mathbf{R}^m(\mathbf{C}^m)$ satisfying

$$\mathbf{e}_i^2 = -1 \quad \text{and} \quad \mathbf{e}_j\mathbf{e}_i = -\mathbf{e}_i\mathbf{e}_j, \quad i \neq j, \quad (i, j = 1, 2, \cdots, m).$$

The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, denoted by $Cl_{0,\ m}$, is the associative algebra over the real (complex) field $\mathbf{R}$ ($\mathbf{C}$). Clearly, it is non-commutative. A general element in $Cl_{0,\ m}$, therefore, is of the form $x = \sum_S x_S \mathbf{e}_S$, $x_S \in \mathbf{R}$ ($\mathbf{C}$) and $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}$, and S runs over all the ordered subsets of $\{1, 2, \ldots, m\}$, namely $S = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}$, $1 \le l \le m$. The conjugation of $\mathbf{e}_S$ is defined by $\overline{\mathbf{e}}_S := \overline{\mathbf{e}}_{il} \cdots \overline{\mathbf{e}}_{i1}, \overline{\mathbf{e}}_j = -\mathbf{e}_j$. So the Clifford conjugate of a vector $\underline{x} \in \mathbf{R}^m$ is $\overline{\underline{x}} = -\underline{x}$, while the Clifford conjugate of a complex vector $\underline{x} \in \mathbf{C}^m$ is $\overline{\underline{x}} = -\overline{x_1}\mathbf{e}_1 - \cdots - \overline{x_m}\mathbf{e}_m$.

The real (complex) *paravector space* $\mathbf{R}^{m+1}(\mathbf{C}^{m+1})$ is the linear subspace defined by

$$\mathbf{R}^{m+1}(\mathbf{C}^{m+1}) := \text{span}_{\mathbf{R}(\mathbf{C})}\{1, \mathbf{e}_1, \ldots, \mathbf{e}_m\} \subset Cl_{0,m},$$

with elements of the form $x := x_0 + \underline{x}, x_0 \in \mathbf{R}(\mathbf{C})$ and $\underline{x} \in \mathbf{R}^m(\mathbf{C}^m)$. The scalar and vector parts of x, Sc (x) and Vec (x), are defined as the x_0 and $\underline{x}$ terms, respectively. We shall always assume the paravector $0 + 0\mathbf{e}_1 + \cdots + 0\mathbf{e}_m := 0$ to be the neutral element of addition in the sequel.

The multiplication of two complex paravectors $x = x_0 + \underline{x}$, $y = y_0 + \underline{y} \in \mathbf{C}^{m+1}$ is given by

$$xy = (x_0 + \underline{x})(y_0 + \underline{y}) = x_0y_0 + x_0\underline{y} + y_0\underline{x} + \underline{x}\underline{y}.$$

In particular, for $y = \overline{x} = \overline{x}_{\overline{0}} + \overline{\underline{x}} \in \mathbf{C}^{m+1}$, we have

$$\begin{aligned} x\overline{x} &= x_0\overline{x}_{\overline{0}} + x_0\overline{\underline{x}} + \overline{x}_{\overline{0}}\underline{x} + \underline{x}\overline{\underline{x}} \\ &= (x_0\overline{x}_{\overline{0}} + \underline{x} \cdot \overline{\underline{x}}) + (x_0\overline{\underline{x}} + \overline{x}_{\overline{0}}\underline{x}) + \underline{x} \wedge \overline{\underline{x}}, \end{aligned}$$

where $\underline{x} \cdot \overline{\underline{x}} = x_1\overline{x_1} + x_2\overline{x_2} + \cdots + x_m\overline{x_m}$ and $\underline{x} \wedge \overline{\underline{x}} = \sum_{k \neq j} x_k \overline{x}_{\overline{j}} \mathbf{e}_k \overline{\mathbf{e}}_{\overline{j}}$. There are three parts, the scalar part $x_0\overline{x}_{\overline{0}} + \overline{\underline{x}} \cdot \underline{x}$, the vector part $x_0\underline{x} + \overline{x}_{\overline{0}}\underline{x}$ and the bi-vector part $\underline{x} \wedge \overline{\underline{x}}$.

Unlike in the complex case, the norm $|x|$ of $x \in \mathbf{C}^{m+1}$ is defined by

$$|x| = \sqrt{\text{Sc}(x\overline{x})} = \sqrt{\text{Sc}(\overline{x}x)} = \sqrt{|x_0|^2 + |x_1|^2 + \cdots + |x_m|^2}.$$

Let us make a notation convention. We say that

$$f : \mathbf{R}^m \to \mathbf{C}^{m+1}, \quad f(x) = [f(x)]_0 + [f(x)]_1\mathbf{e}_1 + \cdots + [f(x)]_m\mathbf{e}_m$$

ia a *complex paravector-valued* function or, briefly, an $\mathbf{C}^{m+1}$-valued function, where the components $[f]_l$ ($l=0, 1,\ldots,m$) are complex-valued functions defined in $\mathbf{R}^m$. Continuity, differentiability, integrability, and so on, which are ascribed to f are defined componentwise. We will work with both the real- (respectively, complex-)linear Hilbert space of square integrable $\mathbf{R}^{m+1}$- ($\mathbf{C}^{m+1}$-)valued functions defined in $\mathbf{R}^m$, that we denote by $L^2(\mathbf{R}^m; \mathbf{R}^{m+1})$ ($L^2(\mathbf{R}^m; \mathbf{C}^{m+1})$). In this assignment, the inner products are defined by

$$\langle f, g \rangle_{L^2(\mathbf{R}^m;\mathbf{R}^{m+1})} := \int_{\mathbf{R}^m} f(\underline{x})\overline{g(\underline{x})}\ d\underline{x}$$

and

$$\langle f, g \rangle_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})} := \int_{\mathbf{R}^m} \text{Sc}(f(\underline{x})\overline{g(\underline{x})})\ d\underline{x}. \tag{2}$$

Remark 2.1. We use different inner products for functions in $L^2(\mathbf{R}^m; \mathbf{R}^{m+1})$ and $L^2(\mathbf{R}^m; \mathbf{C}^{m+1})$. The reason is that, for the real paravector-valued function f, the product $f\overline{f}$ is real-valued. Then the norm can be defined as the usual L^2 norm as $\|f\|_{L^2(\mathbf{R}^m;\mathbf{R}^{m+1})} = \langle f, f \rangle$. While, for the complex paravector-valued function f, the product $f\overline{f}$ in general is no longer real-valued. In order to satisfy the properties of norm, we adapt the scalar part to define the inner product in $L^2(\mathbf{R}^m; \mathbf{C}^{m+1})$.

Denote the module by

$$L^1(\mathbf{R}^m; \mathbf{C}^{m+1}) := \{f : \mathbf{R}^m \to \mathbf{C}^{m+1} | \times \|f\|_{L^1(\mathbf{R}^m;\mathbf{C}^{m+1})} = \int_{\mathbf{R}^m} |f(\underline{x})|\ d\underline{x} < \infty\}.$$

If $f \in L^1(\mathbf{R}^m; \mathbf{C}^{m+1})$, then the *Clifford Fourier transform* of f is defined by

$$F\{f\}(\underline{\xi}) := \frac{1}{\sqrt{(2\pi)^m}} \int_{\mathbf{R}^m} f(\underline{x}) e^{-\mathbf{i}\langle \underline{x}, \underline{\xi} \rangle d\underline{x}} \tag{3}$$

where $\langle \underline{x}, \underline{\xi} \rangle := x_1\xi_1 + \cdots + x_m\xi_m$ is the usual inner product in Euclidean space $\mathbf{R}^m$. The well-known Plancherel theorem for Fourier transform of square integrable signals $f \in L^2(\mathbf{R}^m; \mathbf{C}^{m+1})$ holds

$$\int_{\mathbf{R}^m} \text{Sc}(f(\underline{x})\overline{g(\underline{x})})\ d\underline{x} = \int_{\mathbf{R}^m} \text{Sc}(F\{f\}(\underline{\xi})\overline{F\{g\}(\underline{\xi})})\ d\underline{\xi}.$$

If f=g, the Parseval theorem is obtained:

$$\int_{\mathbf{R}^m} |f(\underline{x})|^2 d\underline{x} = \int_{\mathbf{R}^m} |F\{f\}(\underline{\xi})|^2 d\underline{\xi}. \tag{4}$$

To simplify matters further we shall study the unit energy $\mathbf{C}^{m+1}$-valued signals, that is $\|f\|_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})} = 1$.

Remark 2.2. In [26], the (real) Clifford Fourier transform has an isomorphism with the classical Fourier transform (3) in Euclidean space $\mathbf{R}^m$. The reason is that its Clifford Fourier kernel $e^{-i_n\langle \underline{x}, \underline{\xi} \rangle}$ ($i_n = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_m$) can commute with every element of $Cl_{0,m}, m = 3 (\text{mod } 4)$. The Clifford Fourier transform [26] maps real Clifford-valued space signals into real Clifford-valued frequency signals. While the Fourier transform (3) maps real Clifford-valued space signals into *complex* Clifford-valued frequency signals.

3. Linear canonical transforms of hypercomplex signals

The LCT, being a classical object of analysis, occupies a special place in signal processing and optics [57,58,49]. It has more degrees of freedom and is more flexible than the FT and the FrFT, but with similar computation cost as the conventional FT [24,31]. Due to the mentioned advantages, it is natural to extend it to higher dimensional spaces. Higher dimensional extensions of the LCT has been studied in [33,34]. We use the special case of the LCT in [34].

Definition 3.1. Let

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{R}^{2\times 2}$$

be a matrix parameter such that $det(A)=1$. The LCTs of hypercomplex signals $f \in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$ are defined by

$$L\{f\}(\underline{u}):=\begin{cases} \frac{1}{\sqrt{(2\pi)^m b\imath}}\int_{\mathbf{R}^m} f(\underline{x}) e^{\mathbf{i}((a/2b)\underline{x}^2-(1/b)\langle \underline{x},\underline{u}\rangle+(d/2b)\underline{u}^2)} d\underline{x}; & b\neq 0;\\ \sqrt{d}e^{\mathbf{i}(cd/2)\underline{u}^2} f(d\underline{u}); & b=0.\end{cases}$$

and, if $L\{f\}\in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, the inverse LCTs are defined by

$$L^{-1}\{f\}(\underline{u}):=\begin{cases} \frac{1}{\sqrt{-(2\pi)^m b\imath}}\int_{\mathbf{R}^m} f(\underline{x}) e^{-\mathbf{i}((a/2b)\underline{x}^2-(1/b)\langle \underline{x},\underline{u}\rangle+(d/2b)\underline{u}^2)} d\underline{x}; & b\neq 0;\\ \sqrt{d}e^{\mathbf{i}(cd/2)\underline{u}^2} f(d\underline{u}); & b=0.\end{cases}$$

Note that for $b=0$ the LCT of a signal is essentially a chirp multiplication and it is of no particular interest for our objective in this work. Hence, without loss of generality, we set $b\neq 0$ in the following sections unless stated. Therefore

$$L\{f\}(\underline{u})=\int_{\mathbf{R}^m} f(\underline{x})K(\underline{x},\underline{u})\, d\underline{x} \tag{5}$$

with the kernel function

$$K(\underline{x},\underline{u}):=\frac{1}{\sqrt{(2\pi)^m b\imath}} e^{\mathbf{i}((a/2b)\underline{x}^2-(1/b)\langle \underline{x},\underline{u}\rangle+(d/2b)\underline{u}^2)}\, d\underline{x},\quad b\neq 0. \tag{6}$$

It is significant to note that when $a=d=0$, $b=1$, the LCT of f reduces to the FT of f in $\mathbf{R}^m$. That is

$$L\{f\}(\underline{u})=\frac{1}{\sqrt{(2\pi)^m \mathbf{i}}}\int_{\mathbf{R}^m} f(\underline{x})e^{-\mathbf{i}\langle \underline{x},\underline{u}\rangle}\, d\underline{x}=\frac{1}{\sqrt{\mathbf{i}}}F\{f\}(\underline{u}),$$

where $F\{f\}$ is the FT of f given by (3).

Furthermore, they have the following close relationship.

Lemma 3.1. *Let*

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{R}^{2\times 2}$$

be a matrix parameter such that $det(A)=1$ *and* $b\neq 0$. *Let* $f\in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, *then*

$$L\{f\}(\underline{u})=\frac{1}{\sqrt{b\imath}} e^{\mathbf{i}(d/2b)\underline{u}^2} F\left\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}\left(\frac{\underline{u}}{b}\right). \tag{7}$$

Proof. By the definition of $L\{f\}$ in (5), a direct computation shows that

$$\begin{aligned} L\{f\}(\underline{u}) &= \int_{\mathbf{R}^m} f(\underline{x})K(\underline{x},\underline{u})\, d\underline{x}\\ &= \frac{1}{\sqrt{(2\pi)^m b\imath}}\int_{\mathbf{R}^m} f(\underline{x}) e^{\mathbf{i}((a/2b)\underline{x}^2-(1/b)\langle \underline{x},\underline{u}\rangle+(d/2b)\underline{u}^2)}\, d\underline{x}\\ &= \frac{1}{b\imath} e^{\mathbf{i}(d/2b)\underline{u}^2}\frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbf{R}^m}\left(f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right)e^{-\mathbf{i}\langle \underline{x},\underline{u}/b\rangle}\, d\underline{x}\\ &= \frac{1}{\sqrt{b\imath}} e^{\mathbf{i}(d/2b)\underline{u}^2} F\left\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}\left(\frac{\underline{u}}{b}\right).\end{aligned}$$

Therefore the result (7) follows. □

Remark 3.1. Since the classical FT (3) is the special case of LCT (5) and the (real) Clifford FT [26] has an isomorphism with the FT (3), the present definition of LCT (5) generalizes the (real) Clifford FT [26].

We then establish the Plancherel theorem for LCT of hypercomplex signals.

Theorem 3.1 (*Plancherel theorem*). *If* $f_1,f_2\in L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$, *then*

$$\langle f_1,f_2\rangle_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})}=\langle L\{f_1\},L\{f_2\}\rangle_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})}.$$

Particularly, if $f_1=f_2=f$, *then the Parseval theorem is obtained. That is*

$$\|f\|_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})}=\|L\{f\}\|_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})}.$$

Proof. By the inner product (2) and (7) in Lemma 3.1, we have

$$\begin{aligned} \langle L\{f_1\},L\{f_2\}\rangle_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})} &= \int_{\mathbf{R}^m} Sc\left(L\{f_1\}(\underline{u})\overline{L\{f_2\}(\underline{u})}\right) d\underline{u}\\ &= \int_{\mathbf{R}^m} Sc\left(\left[\frac{1}{\sqrt{b\mathbf{i}}}e^{\mathbf{i}(d/2b)\underline{u}^2}F\left\{f_1(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}\left(\frac{\underline{u}}{b}\right)\right]\right.\\ &\quad\left.\times\overline{\left[\frac{1}{\sqrt{b\mathbf{i}}}e^{\mathbf{i}(d/2b)\underline{u}^2}F\{f_2(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\}\left(\frac{\underline{u}}{b}\right)\right]}\right) d\underline{u}.\end{aligned}$$

Since the complex value $(1/\sqrt{b\mathbf{i}})e^{\mathbf{i}(a/2b)\underline{x}^2}$ computes with any paravector-valued signals, the above becomes

$$\begin{aligned} &\langle L\{f_1\},L\{f_2\}\rangle_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})}\\ &= \int_{\mathbf{R}^m} Sc\left(F\left\{f_1(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}\left(\frac{\underline{u}}{b}\right)\overline{F\{f_2(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\}\left(\frac{\underline{u}}{b}\right)}\right) d\left(\frac{\underline{u}}{b}\right)\\ &= \int_{\mathbf{R}^m} Sc\left([f_1(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]\overline{[f_2(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]}\right) d\underline{x}\\ &= \int_{\mathbf{R}^m} Sc\left(f_1(\underline{x})\overline{f_2(\underline{x})}\right) d\underline{x}=\langle f_1,f_2\rangle_{L^2(\mathbf{R}^m;\mathbf{C}^{m+1})},\end{aligned}$$

where we have used the Plancherel theorem of Fourier transforms of f in $L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$. □

Theorem 3.1 shows that the total signal energy computed in the time domain equals to the total signal energy in the frequency domain, and the change of domains for convenience of computation.

To proceed with, we prove the following partial derivative properties.

Lemma 3.2. *For* $k=1,\ldots,m$, *if* f *and* $\partial f/\partial x_k\in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, *then*

$$F\left\{\frac{\partial}{\partial x_k}[f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]\right\}(\underline{u})=\mathbf{i}u_kF\left\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}(\underline{u}). \tag{8}$$

Proof. Applying the integration by parts and complex value $-\mathbf{i}u_k$ computes with any paravector-valued signals, we obtain

$$\begin{aligned} &F\left\{\frac{\partial}{\partial x_k}[f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]\right\}(\underline{u})\\ &= \frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbf{R}^m}\frac{\partial}{\partial x_k}[f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]e^{-\mathbf{i}\langle \underline{x},\underline{u}\rangle}\, d\underline{x}\\ &= -\frac{1}{\sqrt{(2\pi)^m}}\int_{\mathbf{R}^m}(-\mathbf{i}u_k)f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}e^{-\mathbf{i}\langle \underline{x},\underline{u}\rangle}\, d\underline{x}\end{aligned}$$

$$= \mathbf{i}u_k F\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\}(\underline{u}). \qquad \square$$

Lemma 3.3. *For $k = 1,\ldots,m$, suppose $f \in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, $u_k L\{f\}$ $x_k f$ and $\partial f/\partial x_k \in L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$, then*

$$\int_{\mathbf{R}^m} u_k^2 \left| L\{f\}(\underline{u})\right|^2 \, d\underline{u} = b^2 \int_{\mathbf{R}^m} \left|\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right|^2 \, d\underline{x}. \tag{9}$$

Proof. By (7) in Lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbf{R}^m} u_k^2 |L\{f\}(\underline{u})|^2 \, d\underline{u} \\
&\quad= \int_{\mathbf{R}^m} u_k^2 \left|\frac{1}{\sqrt{b\mathbf{i}}} e^{\mathbf{i}(d/2b)\underline{u}^2} F\left\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}\left(\frac{\underline{u}}{b}\right)\right|^2 d\underline{u} \\
&\quad= b^2 \int_{\mathbf{R}^m} \left|\frac{u_k}{b} e^{\mathbf{i}(d/2b)\underline{u}^2} F\left\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\right\}\left(\frac{\underline{u}}{b}\right)\right|^2 d\left(\frac{\underline{u}}{b}\right) \\
&\quad= b^2 \int_{\mathbf{R}^m} |t_k F\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\}(\underline{t})|^2 \, d\underline{t} \\
&\quad= b^2 \int_{\mathbf{R}^m} |\mathbf{i}t_k F\{f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}\}(\underline{t})|^2 \, d\underline{t}.
\end{aligned}$$

Applying (8) in Lemma 3.2 and Parseval theorem of Fourier transform of $\partial/\partial x_k [f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]$, we obtain

$$\begin{aligned}
\int_{\mathbf{R}^m} u_k^2 |L\{f\}(\underline{u})|^2 \, d\underline{u} &= b^2 \int_{\mathbf{R}^m} \left|F\left\{\frac{\partial}{\partial x_k}[f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]\right\}(\underline{t})\right|^2 d\underline{t} \\
&= b^2 \int_{\mathbf{R}^m} \left|\frac{\partial}{\partial x_k}[f(\underline{x})e^{\mathbf{i}(a/2b)\underline{x}^2}]\right|^2 d\underline{x} \\
&= b^2 \int_{\mathbf{R}^m} \left|\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right|^2 d\underline{x}. \qquad \square
\end{aligned}$$

Lemma 3.4. *If f and $g \in L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$, the Schwarz's inequality holds*

$$\frac{1}{4}\left\{\int_{\mathbf{R}^m} \mathrm{Sc}\left(f\overline{g} + g\overline{f}\right) d\underline{x}\right\}^2 \le \|f\|^2 \|g\|^2. \tag{10}$$

Proof. For any real number ε, we have

$$\begin{aligned}
0 &\le \|f + \varepsilon g\|^2 \\
&= \int_{\mathbf{R}^m} \mathrm{Sc}\left((f+\varepsilon g)\overline{(f+\varepsilon g)}\right) d\underline{x} \\
&= \int_{\mathbf{R}^m} \mathrm{Sc}(f\overline{f}) \, d\underline{x} + \varepsilon \int_{\mathbf{R}^m} \mathrm{Sc}\left(f\overline{g} + g\overline{f}\right) d\underline{x} + \varepsilon^2 \int_{\mathbf{R}^m} \mathrm{Sc}(g\overline{g}) \, d\underline{x}.
\end{aligned}$$

Then we have

$$\left\{\int_{\mathbf{R}^m} \mathrm{Sc}(f\overline{g} + g\overline{f}) \, d\underline{x}\right\}^2 - 4\left(\int_{\mathbf{R}^m} \mathrm{Sc}\left(f\overline{f}\right) d\underline{x}\right)\left(\int_{\mathbf{R}^m} \mathrm{Sc}(g\overline{g}) \, d\underline{x}\right) \le 0.$$

Then

$$\frac{1}{4}\left\{\int_{\mathbf{R}^m} \mathrm{Sc}(f\overline{g} + g\overline{f}) \, d\underline{x}\right\}^2 \le \|f\|^2 \|g\|^2.$$

This completes the proof of (10). □

Remark 3.2. For the complex-valued functions f and g, we have

$$\frac{1}{4}\left\{\int_{\mathbf{R}^m} \mathrm{Sc}(f\overline{g} + g\overline{f}) d\underline{x}\right\}^2 = \frac{1}{4}\left\{\int_{\mathbf{R}^m} \mathrm{Sc}(f\overline{g} + \overline{f\overline{g}}) \, d\underline{x}\right\}^2$$

$$= \left\{\int_{\mathbf{R}^m} \mathrm{Sc}(f\overline{g}) \, d\underline{x}\right\}^2 \le \|f\|^2 \|g\|^2.$$

That is

$$\mathrm{Sc}(f\overline{g} + g\overline{f}) = 2\mathrm{Sc}(f\overline{g}).$$

While in the complex paravector-valued functions f and g, we have

$$\mathrm{Sc}(f\overline{g} + g\overline{f}) \ne 2\mathrm{Sc}(f\overline{g}).$$

We are now in the heart of the matter.

4. Uncertainty principles

In signal processing much effort has been placed in the study of the classical Heisenberg uncertainty principle during the last years. To our knowledge, a systematic work on the investigation of uncertainty relations using the LCT of hypercomplex signal is not carried out.

In the following we explicitly prove and generalize the classical uncertainty principle to Clifford module functions using the LCTs. We also give an explicit proof for Gaussian signals (Gabor filters) to be indeed the only signals that minimize the uncertainty.

First, we assume signals that have zero mean spaces:

$$0 = \langle x_k \rangle := \int_{\mathbf{R}^m} x_k |f(\underline{x})|^2 \, d\underline{x} \tag{11}$$

and

$$0 = \langle \underline{x} \rangle := \int_{\mathbf{R}^m} \underline{x} |f(\underline{x})|^2 \, d\underline{x}. \tag{12}$$

Furthermore, zero mean frequencies:

$$0 = \langle u_k \rangle := \int_{\mathbf{R}^m} u_k |L\{f\}(\underline{u})|^2 \, d\underline{u} \tag{13}$$

and

$$0 = \langle \underline{u} \rangle := \int_{\mathbf{R}^m} \underline{u} |L\{f\}(\underline{u})|^2 \, d\underline{u} \tag{14}$$

Let us now begin the proofs of two uncertainty relations.

Theorem 4.1. *For $k = 1,\ldots,m$, suppose unit energy signal $f \in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, $u_k L\{f\}$, $x_k f$ and $\partial f/\partial x_k \in L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$, then*

$$\left(\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} u_k^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \ge \frac{b^2}{4} \tag{15}$$

and equality is achieved when

$$f(\underline{x}) = C_0 \exp\left(-\frac{1}{2}(\alpha_1 x_1^2 + \cdots + \alpha_m x_m^2)\right), \tag{16}$$

where α_k are positive real constants and $C_0 = (\alpha_1 \cdots \alpha_m/\pi^2)^{1/4}$.

Proof. Applying (9) in Lemma 3.3, and Schwarz's inequality (10), we have

$$\begin{aligned}
&\left(\int_{\mathbf{R}^m} x_k^2 |f(\underline{x})|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} u_k^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \\
&\quad= b^2 \left(\int_{\mathbf{R}^m} \left|x_k f(\underline{x})\right|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} \left|\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right|^2 \, d\underline{x}\right)
\end{aligned}$$

$$\geq \frac{b^2}{4}\left|\int_{\mathbf{R}^m} \mathrm{Sc}\left(x_k f(\underline{x})\overline{\left[\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right]} + \left[\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right]\overline{x_k f(\underline{x})}\right) d\underline{x}\right|^2. \tag{17}$$

By direct calculation,

$$\begin{aligned}&\left|\int_{\mathbf{R}^m} \mathrm{Sc}\left(x_k f(\underline{x})\overline{\left[\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right]} + \left[\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right]\overline{x_k f(\underline{x})}\right) d\underline{x}\right| \\ &= \left|\int_{\mathbf{R}^m} \mathrm{Sc}\left(\left[f(\underline{x})\frac{\partial}{\partial x_k}(\overline{f(\underline{x})})x_k + \mathbf{i}\frac{a}{b} x_k^2 f(\underline{x})\overline{f(\underline{x})}\right] + \left[\frac{\partial}{\partial x_k}(f(\underline{x}))\overline{f(\underline{x})}x_k - \mathbf{i}\frac{a}{b} x_k^2 f(\underline{x})\overline{f(\underline{x})}\right]\right) d\underline{x}\right| \\ &= \left|\int_{\mathbf{R}^m} \mathrm{Sc}\left(\frac{\partial}{\partial x_k}(x_k f(\underline{x})\overline{f(\underline{x})} d\underline{x}) - \int_{\mathbf{R}^m} |f(\underline{x})|^2\right) d\underline{x}\right| \\ &= 1.\end{aligned} \tag{18}$$

The first term of (18) is a perfect differential and integrates to zero. The second term gives one half since we assume the signal is unit energy. Therefore we have the result (15) follows.

Since the minimum value for the uncertainty product is $b^{2/4}$, we can ask what signals have that minimum value. The Schwarz's inequality (17) becomes an equality when the two functions are proportional to each other. Hence we take

$$-Cx_k f(\underline{x}) = \frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x}),$$

where C is an arbitrary constant. Solving this differential equation, we get

$$f(\underline{x}) = C_0 \exp\left(-\frac{1}{2}\left[\left(C_1 - \mathbf{i}\frac{a}{b}\right)x_1^2 + \cdots + \left(C_n - \mathbf{i}\frac{a}{b}\right)x_n^2\right]\right),$$

where C_0 is a constant of integration. In order to make $f \in L^1(\mathbf{R}^m; \mathbf{C}^{m+1})$, C_k can be chosen such that

$$\alpha_k := C_k - \mathbf{i}\frac{a}{b},$$

where α_k is positive real constant.

$$f(\underline{x}) = C_0 \exp(-\tfrac{1}{2}(\alpha_1 x_1^2 + \cdots + \alpha_m x_m^2)), \tag{19}$$

The value of C_0 can be found out by noting that f must be unit norm. That is

$$C_0 := \left(\frac{\alpha_1 \cdots \alpha_m}{\pi^2}\right)^{1/4}.$$

Thus, $f(\underline{x})$ given in (19), which satisfies the Cauchy–Schwarz inequality (17), turns out to be a Gaussian function and the theorem is proved. □

Since the Gaussian function $f(\underline{x})$ of (19) achieves the minimum width-bandwidth product, it is theoretically a very good prototype wave form. One can therefore construct a basic wave form using spatially or frequency scaled versions of $f(\underline{x})$ to provide multiscale spectral resolution. In quaternion analysis, such a wavelet basis construction derived from a Gaussian function prototype wave form has been realized, for example, in the quaternionic wavelet transforms in [4]. The optimal space-frequency localization is also another reason why 2D Clifford–Gabor bandpass filters were suggested in [8].

We now derive a new directional uncertainty principle for complex paravector-valued signals subject to the linear canonical transformation.

Theorem 4.2. *For $k = 1, \ldots, m$, suppose $f \in L^1(\mathbf{R}^m; \mathbf{C}^{m+1})$, $u_k L\{f\}$, $x_k f$ and $\partial f/\partial x_k \in L^2(\mathbf{R}^m; \mathbf{C}^{m+1})$, then*

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{u}|^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \geq \frac{b^2 m^2}{4}. \tag{20}$$

and equality is achieved when f is a Gaussian function, i.e.,

$$f(\underline{x}) = C_0 \exp\left(-\frac{1}{2}(\alpha_1 x_1^2 + \cdots + \alpha_m x_m^2)\right), \tag{21}$$

where α_k are positive real constants and $C_0 = (\alpha_1 \cdots \alpha_m/\pi^2)^{1/4}$.

Proof. Applying (9) in Lemma 3.3, we have

$$\begin{aligned}&\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{u}|^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \\ &= \left(\int_{\mathbf{R}^m} \sum_{k=1}^m x_k^2 |f(\underline{x})|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} \sum_{j=1}^m u_j^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \\ &= b^2 \int_{\mathbf{R}^m} \left(\sum_{k=1}^m |x_k|^2 |f(\underline{x})|^2\right) d\underline{x} \int_{\mathbf{R}^m} \left(\sum_{j=1}^m \left|\frac{\partial}{\partial x_j} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_j f(\underline{x})\right|^2\right) d\underline{x} \\ &\geq b^2 \left|\int_{\mathbf{R}^m} \left(\sum_{k=1}^m |x_k|^2 |f(\underline{x})|^2\right)^{1/2} \left(\sum_{j=1}^m \left|\frac{\partial}{\partial x_j} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_j f(\underline{x})\right|^2\right)^{1/2} d\underline{x}\right|^2\end{aligned} \tag{22}$$

where we have used the Schwarz inequality of square integrable real value signals. For $i = 1, \ldots, m$, the Schwarz inequality of paravector-numbers s_i and t_i is given by

$$\left|\frac{1}{2}\sum_{i=1}^m Sc(s_i \overline{t_i} + t_i \overline{s_i})\right|^2 \leq \left(\sum_{j=1}^m |s_j|^2\right)\left(\sum_{k=1}^m |t_k|^2\right). \tag{23}$$

Equality holds if and only if s_k and t_k are linearly dependent. One has

$$\begin{aligned}&\left(\sum_{k=1}^m x_k^2 |f(\underline{x})|^2\right)^{1/2}\left(\sum_{j=1}^m \left|\frac{\partial}{\partial x_j} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_j f(\underline{x})\right|^2\right)^{1/2} \\ &\geq \frac{1}{2}\sum_{k=1}^m Sc\left(x_k f(\underline{x})\overline{\left(\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right)} + \left(\frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x})\right)\overline{(x_k f(\underline{x}))}\right) \\ &= \frac{1}{2}\sum_{k=1}^m Sc\left(x_k f(\underline{x})\frac{\partial}{\partial x_k}\left(\overline{f(\underline{x})}\right) + \frac{\partial}{\partial x_k}(f(\underline{x}))x_k \overline{f(\underline{x})}\right) \\ &= \frac{1}{2}\sum_{k=1}^m Sc\left(\frac{\partial}{\partial x_k}(x_k f(\underline{x})\overline{f(\underline{x})}) - f(\underline{x})\overline{f(\underline{x})}\right).\end{aligned} \tag{24}$$

Therefore (22) becomes

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 d\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{u}|^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \geq \frac{b^2 m^2}{4}.$$

We finally show that equality in (22) and (24) are satisfied if and only if f is a Gaussian function given by (21).

Schwarz's inequality (10) of real valued signals becomes an equality when the two functions are proportional to each other. Hence we have

$$-C_k x_k f(\underline{x}) = \frac{\partial}{\partial x_k} f(\underline{x}) - \mathbf{i}\frac{a}{b} x_k f(\underline{x}),$$

where C_k is an arbitrary constant. Solving this differential equation, we get

$$f(\underline{x}) = C_0 \exp\left(-\frac{1}{2}(\alpha_1 x_1^2 + \cdots + \alpha_m x_m^2)\right),$$

where α_k are positive real constant and

$$C_0 := \left(\frac{\alpha_1 \cdots \alpha_m}{\pi^2}\right)^{1/4}.$$

Thus, $f(\underline{x})$ given by (21), which satisfies the Cauchy–Schwarz inequality (17), turns out to be a Gaussian function and the theorem is proved. □

When $a=0$, $b=1$, we have

Corollary 4.1. *For $k=1,2,\ldots,m$, if $f \in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, $x_k f$ and $\partial f/\partial x_k \in L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$, then*

$$\left(\int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \, d\underline{x}\right)\left(\int_{\mathbf{R}^m} |\underline{u}|^2 |F\{f\}(\underline{u})|^2 \, d\underline{u}\right) \ge \frac{m^2}{4},$$

where $F\{f\}$ is the Fourier transform of f given by (3).

Furthermore when $m=1$, this reduces to the classical uncertainty principle [10] for complex-valued signal.

Corollary 4.2. *For a complex valued signal $f \in L^1(\mathbf{R};\mathbf{C})$, if f, f' and $xf(x) \in L^2(\mathbf{R};\mathbf{C})$, then*

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx\right)\left(\int_{-\infty}^{\infty} u^2 |F\{f\}(u)|^2 \, d(u)\right) \ge \frac{1}{4}.$$

Remark 4.1.

(i) Note that these uncertainty relations (Theorems 4.1 and 4.2) are consistent for the complex-valued signals ($m=1$). While Theorem 4.2 formulates a directional uncertainty principle for complex paravector-valued signals and Theorem 4.1 gives the spatial case.

(ii) Note that many studies of uncertainty principles in the Clifford algebra, most of the researches cannot be reduced to the 1D ($m=1$) case, for instance [26,27,41,42].

Finally we prove that the standard deviation does not depend on the mean because it is defined as the broadness about the mean. We then obtain the general results for the uncertainty relations.

Theorem 4.3. *For $k=1,\ldots,m$, suppose unit energy signal $f \in L^1(\mathbf{R}^m;\mathbf{C}^{m+1})$, $u_k L\{f\}$, $x_k f$ and $\partial f/\partial x_k \in L^2(\mathbf{R}^m;\mathbf{C}^{m+1})$, then we have*

(i)
$$\left(\int_{\mathbf{R}^m} (x_k - \langle x_k\rangle)^2 |f(\underline{x})|^2 \, d\underline{x}\right) \times \left(\int_{\mathbf{R}^m} (u_k - \langle u_k\rangle)^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \ge \frac{b^2}{4},$$

(ii)
$$\left(\int_{\mathbf{R}^m} |\underline{x} - \langle \underline{x} > |^2 |f(\underline{x})|^2 \, d\underline{x}\right) \times \left(\int_{\mathbf{R}^m} |\underline{u} - \langle \underline{u}\rangle|^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}\right) \ge \frac{b^2 m^2}{4}.$$

Furthermore, equalities of (i) and (ii) are achieved when

$$f(\underline{x}) = C_0 \exp\left(-\frac{1}{2}(\alpha_1 x_1^2 + \cdots + \alpha_m x_m^2)\right),$$

where $\langle x_k\rangle, \langle \underline{x}\rangle, \langle u_k\rangle, \langle \underline{u}\rangle$ are defined by (11), (12), (13) and (14), respectively, the constants α_k are positive real numbers and $C_0 = (\alpha_1 \cdots \alpha_m/\pi^2)^{1/4}$.

Proof. The proof of part (i) works very similar to the classical case in the complex plane. We therefore do not repeat it here.

To prove (ii), if we take a signal f that has zero mean space (12) and zero mean frequency (14), we want a signal of the same shape but with particular mean space $\langle \underline{x}\rangle$ and frequency $\langle \underline{u}\rangle$, then a new signal can be defined by

$$g(\underline{x}) = e^{\mathbf{i}(1/b)\langle \underline{x} - \langle \underline{x}\rangle, \langle \underline{u}\rangle\rangle} e^{\mathbf{i}(a/b)\langle \underline{x}, \langle \underline{x}\rangle\rangle} f(\underline{x} - \langle \underline{x}\rangle).$$

Therefore

$$\int_{\mathbf{R}^m} |\underline{x} - \langle \underline{x}\rangle|^2 |g(\underline{x})|^2 \, d\underline{x} = \int_{\mathbf{R}^m} |\underline{x} - \langle \underline{x}\rangle|^2 |f(\underline{x} - \langle \underline{x}\rangle)|^2 \, d\underline{x} = \int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \, d\underline{x},$$

and

$$\int_{\mathbf{R}^m} |\underline{u} - \langle \underline{u}\rangle|^2 |L\{g\}(\underline{u})|^2 \, d\underline{u} = \int_{\mathbf{R}^m} |\underline{u} - \langle \underline{u}\rangle|^2 |L\{f\}(\underline{u} - \langle \underline{u}\rangle)|^2 \, d\underline{u} = \int_{\mathbf{R}^m} |\underline{u}|^2 |L\{f\}(\underline{u})|^2 \, d\underline{u}.$$

Here we use the property $|L\{g\}(\underline{u})| = |L\{f\}(\underline{u} - \langle \underline{u}\rangle)|$. In fact, by Eq. (5), we have

$$\begin{aligned} |L\{g\}(\underline{u})| &= \left|\int_{\mathbf{R}^m} g(\underline{x}) K(\underline{x}, \underline{u}) \, d\underline{x}\right| \\ &= \left|\int_{\mathbf{R}^m} e^{\mathbf{i}(1/b)\langle \underline{x} - \langle \underline{x}\rangle, \langle \underline{u}\rangle\rangle} e^{\mathbf{i}(a/b)\langle \underline{x}, \langle \underline{x}\rangle\rangle} f(\underline{x} - \langle \underline{x}\rangle) K(\underline{x}, \underline{u}) \, d\underline{x}\right| \\ &= \left|\int_{\mathbf{R}^m} f(\underline{x} - \langle \underline{x}\rangle) K(\underline{x} - \langle \underline{x}\rangle, \underline{u} - \langle \underline{u}\rangle) d\underline{x}\right| \\ &= |L\{f\}(\underline{u} - \langle \underline{u}\rangle)|. \end{aligned}$$

Applying Theorem 4.2 to the zero mean space and frequency signal f, it completes the proof. □

5. Example

Consider the complex paravector-valued signal

$$f(\underline{x}) = \frac{s}{|s + \underline{x}|^{m+1}} e^{-\mathbf{i}(a/2b)\underline{x}^2}, \quad s > 0.$$

Obversely, $(1/\|f\|_{L^2}) f(\underline{x})$ is unit energy. By Lemma 3.1, we have

$$L\{f\}(\underline{u}) = \frac{1}{\sqrt{b\imath}} e^{\mathbf{i}(d/2b)\underline{u}^2} F\left\{\frac{s}{|s + \underline{x}|^{m+1}}\right\}\left(\frac{\underline{u}}{b}\right).$$

Direct calculation shows that

$$F\left\{\frac{s}{|s + \underline{x}|^{m+1}}\right\}(\underline{u}) = \frac{\sqrt{\pi}}{\sqrt{2^m}\,\Gamma(\frac{m+1}{2})} e^{-s|\underline{u}|}.$$

Moreover

$$\frac{1}{\|f\|_{L^2}^2} \int_{\mathbf{R}^m} |\underline{x}|^2 |f(\underline{x})|^2 \, d\underline{x} = \frac{1}{\|f\|_{L^2}^2} \int_{\mathbf{R}^m} \frac{s^2 |\underline{x}|^2}{|s + \underline{x}|^{2m+2}} \, d\underline{x}$$

$$=\frac{\int_{\mathbf{R}^m}\frac{|\underline{x}|^2}{|s+\underline{x}|^{2m+2}}\,d\underline{x}}{\int_{\mathbf{R}^m}\frac{1}{|s+\underline{x}|^{2m+2}}\,d\underline{x}}=s^2$$

and

$$\frac{1}{\|f\|_{L^2}^2}\int_{\mathbf{R}^m}|\underline{u}|^2|L\{f\}(\underline{u})|^2\,d\underline{u}=\frac{1}{\|L\{f\}\|_{L^2}^2}\int_{\mathbf{R}^m}|\underline{u}|^2|L\{f\}(\underline{u})|^2\,d\underline{u}$$

$$=\frac{b^2\int_{\mathbf{R}^m}|\underline{t}|^2e^{-2s|\underline{t}|}\,d\underline{t}}{\int_{\mathbf{R}^m}e^{-2s|\underline{t}|}\,d\underline{t}}=\frac{b^2}{4s^2}\frac{\Gamma(m+2)}{\Gamma(m)}=\frac{b^2(m+1)m}{4s^2}.$$

Therefore

$$\left(\frac{1}{\|f\|_{L^2}^2}\int_{\mathbf{R}^m}|\underline{x}|^2|f(\underline{x})|^2\,d\underline{x}\right)\left(\frac{1}{\|f\|_{L^2}^2}\int_{\mathbf{R}^m}|\underline{u}|^2|L\{f\}(\underline{u})|^2\,d\underline{u}\right)$$

$$=b^2\frac{m(m+1)}{4}\geq\frac{b^2m^2}{4}.$$

Thus the unit energy complex paravector-signal f is indeed satisfied the uncertainty Theorem 4.2.

6. Conclusion

In this paper we developed the definition of Clifford LCT. The various properties of LCT such as partial derivative, Plancherel and Parseval theorems are established. Using the well-known Plancherel theorem, we established two uncertainty principles for hypercomplex signals in the LCT domains. The first principle (Theorem 4.1) states that the product of the variances of multivector- (complex paravector-)valued signals in the spatial and frequency domains has a lower bound. While the second principle (Theorem 4.2) states the directional case. It is shown that only a m-D Guassian signal minimizes the uncertainties. With the help of these principles, they are useful in the time–frequency analysis, developing sampling theory in the m-D case, filter design, signal synthesis and optics.

We also note that the present theory can be generalized to complex Clifford- ($Cl_{0,m}$-)valued signals without difficulties. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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