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THE DIFFERENTIAL INTEGRAL EQUATIONS ON SMOTH CLOSED ORIENTABLE MANIFOLDS[1]

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Abstract Using integration by parts and Stokes' formula the authors give a new definition of Hadamard principal value of higher order singular integrals with Bochner-Martinelli kernel on smooth closed orientable manifolds in $\mathbf{C}^n$. The Plemelj formula and composite formula of higher order singular integral are obtained. Differential integral equations on smooth closed orientable manifolds are treated by using the composite formula.

Key words Bochner-Martinelli kernel, Plemelj formula, Composite formula, Higher order singular integral, Differential integral Equation

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1 Introduction

Since the limit value formula, viz. the Plemelj formula, of the Cauchy type integral with Bochner-Martinelli kernel was proved in 1957[1], it has been successfully used to the study of singular integral equations, solving the $\overline{\partial}_b$-equation, holomorphic extension, $\overline{\partial}$-closed extension and $C-R$ manifolds[2−5]. Evidently, the research of higher order singular integrals with Bochner-Martinelli kernel itself also has important significance. In 1952, J. Hadamard first defined the principal value of higher order singular integrals, that is the so called Hadamard principal value, by using which he solved the Cauchy problem of hyperbolic partial differential equation. The idea of Hadamard is to separate its finite part, i.e. the principal value[6], from a divergent integral on real axis. In 1957, C. Fox extended the idea of Hadamard to the case of higher order singular integrals in the complex plane[7]. The calculations of all those principal values of higher order singular integrals are very complicated, and they are very difficult to be extended to the several complex variables case. In 1990 Wang Xiaoqin studied the higher order singular integrals on complex hypersphere in $\mathbf{C}^n$ and the higher order singular integrals with Bochner-Martinelli kernel. She obtained some remarkable results[8] with simpler methods, but the calculations are still complicated. In [9] by using integration by parts and Stokes' formula we

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reduce the order of singularity of higher order singular integrals, and we finally use the Cauchy singular integral to express the higher order singular integrals. The Hadamard principal values therefore are expressed by the Cauchy principal value.

For simplicity, we consider higher order singular integrals on the boundary of a bounded domain D with smooth boundary in the complex plane

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta, \quad z\in D, \tag{1}$$

where $f(\zeta)\in H_1(\alpha)$, that is the set of the functions whose first order derivatives satisfy the Hölder condition of index α. Singular integral (1) under Cauchy principal value is divergent. Using integration by parts and Stokes' formula we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta &= \frac{1}{2\pi i}\int_{\partial D}\left\{d\left[\left(\frac{-1}{\zeta-z}\right)f(\zeta)\right]-\left(\frac{-1}{\zeta-z}\right)df(\zeta)\right\}\\ &= \frac{1}{2\pi i}\int_{\partial D}\frac{f'(\zeta)}{\zeta-z}d\zeta, \quad z\in D.\end{aligned}$$

Therefore the study of the Cauchy type integral $\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta$ is equivalent to the study of the Cauchy type integral $\frac{1}{2\pi i}\int_{\partial D}\frac{f'(\zeta)}{\zeta-z}d\zeta$. Since $f'(\zeta)\in H(\alpha)$, when $z=\eta\in\partial D$, the Cauchy principal value of the right hand side exists. Therefore, if we denote the Hadamard principal value by FP, the Cauchy principal value by PV, then we may define

$$FP\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-\eta)^2}d\zeta = PV\frac{1}{2\pi i}\int_{\partial D}\frac{f'(\zeta}{\zeta-\eta}d\zeta,\ \eta\in\partial D. \tag{2}$$

This definition agrees with the original idea of Hadamard to separate the finite part from the divergent integral. Moreover, in our definition the Hadamard principal value is expressed by the Cauchy principal value that gives us the convenience of utilizing the results of the Cauchy principal value. In the following we will use this idea to study the higher order singular integral with Bochner-Martinelli kernel in $\mathbf{C}^n$.

2 Hadamard Principal Value and Plemelj Formula

Let D be a bounded domain in $\mathbf{C}^n$, its boundary ∂D be a smooth orientable manifold of dimension $2n-1$ of class $C^{(2)}$. If $f\in A_c(D)$, then we have the well-known Bochner-Martinelli integral

$$f(z)=\int_{\partial D}f(\zeta)K(\zeta,z), \quad z\in\mathbf{C}^n\setminus\partial D, \tag{3}$$

where

$$K(\zeta,z)=\Omega(\overline{\zeta}-\overline{z},\zeta-z):=\frac{(n-1)!}{(2\pi i)^n}\ \frac{w'(\overline{\zeta}-\overline{z})\wedge w(\zeta-z)}{(\overline{\zeta}-\overline{z},\zeta-z)^n}, \tag{4}$$

$$\begin{aligned} w'(v): &= \textstyle\sum_{j=1}^n(-1)^{j+1}v_j dv_1\wedge\cdots\wedge d[v_j]\wedge\cdots\wedge dv_n,\\ w(v): &= dv_1\wedge\cdots\wedge dv_n,\\ <v,u>: &= \textstyle\sum_{j=1}^n v_j u_j.\end{aligned} \tag{5}$$

In order to study the higher order singular integral with Bochner-Martinelli kernel, firstly we study the following higher order Bochner-Martinelli type integral

$$\int_{\partial D} f(\zeta)\frac{\overline{\zeta}_k-\overline{z}_k}{|\zeta-z|^2}K(\zeta,z), \qquad z\in \mathbf{C}^n\setminus\partial D, \tag{6}$$

where f is differentiable and belongs to $H_1(\alpha)$ on ∂D. We have

Lemma 1 Let f be differentiable and belong to $H_1(\alpha)$ on $\overline{D}$, then for any $w\in\partial D$ and any ball $B(w,\epsilon)$ of radius ϵ (ϵ is a sufficient small positive number) centred at $w\in\partial D$, we have

$$\begin{aligned}
&\int_{\partial D\setminus B(\omega,\epsilon)} f(\zeta)\frac{\overline{\xi}_k-\overline{\omega}_k}{|\zeta-\omega|^2}K(\zeta,\omega)\\
=\ & -\frac{1}{n}C_n\int_{\partial(\partial D\setminus B(\omega,\epsilon))} f(\xi)\\
&\cdot\frac{\sum_{j=1}^n(-1)^{j-1}(\overline{\xi}_j-\overline{\omega}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-\omega|^{2n}}\\
&+\frac{1}{n}C_n\int_{\partial D\setminus B(\omega,\epsilon)} df(\zeta)\\
&\cdot\frac{\sum_{j=1}^n(-1)^{j-1}(\overline{\xi}_j-\overline{\omega}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-\omega|^{2n}},
\end{aligned}\tag{7}$$

where $C_n=\frac{(n-1)!}{(2\pi i)^n}$.

Proof By integration by parts we have

$$\begin{aligned}
&\int_{\partial D\setminus B(\omega,\epsilon)} f(\zeta)\frac{\overline{\zeta}_k-\overline{\omega}_k}{|\zeta-\omega|^2}K(\zeta,\omega)=-\frac{1}{n}C_n\int_{\partial D\setminus B(\omega,\epsilon)} f(\zeta)\\
&\cdot\left[\frac{\sum_{j=1}^n(-1)^{j-1}(\overline{\zeta}_j-\overline{\omega}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-\omega|^{2n}}\right]\\
=&-\frac{1}{n}C_n\int_{\partial D\setminus B(\omega,\epsilon)}\\
&\cdot d\left[f(\zeta)\frac{\sum_{j=1}^n(-1)^{j-1}(\overline{\zeta}_j-\overline{\omega}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-\omega|^{2n}}\right]\\
&+\frac{1}{n}C_n\int_{\partial D\setminus B(\omega,\epsilon)} df(\zeta)\\
&\cdot\frac{\sum_{j=1}^n(-1)^{j-1}(\overline{\zeta}_j-\overline{\omega}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-\omega|^{2n}}.
\end{aligned}$$

Applying Stokes' formula to the first term of right hand side, (7) follows immediately.

Now we consider the first term on the right hand side of (7). Since the dimension of integration domain is $2n-2$, but the order of singularity of the integrand is $2n-1$, this integral is divergent. Moreover, since $f\in H_1(\alpha)$ on ∂D, the integral of the second term exists in the sense of the Cauchy principal value. By the idea of Hadamard principal value, we have

Definition 1 If f is a function defined and differentiable on ∂D and $f\in H_1(\alpha)$ on ∂D, we define

$$FP\int_{\partial D} f(\zeta)\frac{\overline{\zeta}_k-\overline{\omega}_k}{|\zeta-\omega|^2}K(\zeta,\omega)$$

$$= PV\frac{1}{n}C_n\int_{\partial D} df(\zeta)$$
$$\cdot\frac{\sum_{j=1}^{n}(-1)^{j-1}(\overline{\zeta}_j-\overline{\omega}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-\omega|^{2n}}. \tag{8}$$

Obviously, the above definition of Hadamard principal value is simpler and clearer than that in [8]. This definition has some advantages: 1) it has discarded the divergent part of the higher order singular integral directly, and only kept the finite part. In this way we can avoid complicated calculations in applications; 2) it is defined at a general point $\omega\in\partial D$; and 3) the Hadamard principal value is in terms of the Cauchy principal value, so we can utilize the results of the Cauchy principal value directly.

As example, by the Plemelj formula of the Cauchy singular integral on $\partial D(see[1-4])$, we have

Theorem 1 (The Plemelj formula for higher order singular integral) Under the assumptions in Definition 1, if z approaches $\omega\in\partial D$ from the inner part and the outer part of D, then for the higher order Bochner-Martinelli type integral

$$F(z)=\int_{\partial D} f(\zeta)\frac{\overline{\zeta}_k-\overline{z}_k}{|\zeta-z|^2}K(\zeta,z) \qquad z\in\mathbf{C}^n\setminus\partial D \tag{9}$$

it follows the Plemelj formula

$$F_i(\omega)=FP\int_{\partial D} f(\zeta)\frac{\overline{\zeta}_k-\overline{\omega}_k}{|\zeta-\omega|^2}K(\zeta,z)+\frac{1}{2n}\left[\frac{\partial f}{\partial\omega_k}(\omega)+(-1)^n\frac{\partial f}{\partial\omega_1}(\omega)\right], \tag{10}$$
$$F_e(\omega)=FP\int_{\partial D} f(\zeta)\frac{\overline{\zeta}_k-\overline{\omega}_k}{|\zeta-\omega|^2}K(\zeta,z)-\frac{1}{2n}\left[\frac{\partial f}{\partial\omega_k}(\omega)+(-1)^n\frac{\partial f}{\partial\omega_1}(\omega)\right]. \tag{11}$$

Proof By integration by parts and Stokes' formula we have

$$F(z)=\int_{\partial D} f(\zeta)\frac{\overline{\zeta}_k-\overline{z}_k}{|\zeta-z|^2}K(\zeta,z)$$
$$=-\frac{1}{n}C_n\int_{\partial D} f(\zeta)$$
$$\cdot d\left[\frac{\sum_{j=1}^{n}(-1)^{j-1}(\overline{\zeta}_j-\overline{z}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-z|^{2n}}\right]$$
$$=-\frac{1}{n}C_n\int_{\partial D}$$
$$\cdot d\left[f(\zeta)\frac{\sum_{j=1}^{n}(-1)^{j-1}(\overline{\zeta}_j-\overline{z}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-z|^{2n}}\right]$$
$$+\frac{1}{n}C_n\int_{\partial D} df(\zeta)$$
$$\cdot\frac{\sum_{j=1}^{n}(-1)^{j-1}(\overline{\zeta}_j-\overline{z}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-z|^{2n}}$$
$$=\frac{1}{n}C_n\int_{\partial D} df(\zeta)$$
$$\cdot\frac{\sum_{j=1}^{n}(-1)^{j-1}(\overline{\zeta}_j-\overline{z}_j)(-1)^{k-1}d\zeta_1\wedge\cdots\wedge[d\zeta_k]\wedge\cdots\wedge d\zeta_n\wedge d\overline{\zeta}_1\wedge\cdots\wedge[d\overline{\zeta}_j]\wedge\cdots\wedge d\overline{\zeta}_n}{|\zeta-z|^{2n}}.$$

Therefore,

$$F_i(\omega) = \lim_{z\to\omega^+} \frac{1}{n} C_n \int_{\partial D} df(\zeta) \cdot \frac{\sum_{j=1}^n (-1)^{j-1}(\overline{\zeta}_j - \overline{z}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge [d\overline{\zeta}_j] \wedge \cdots \wedge d\overline{\zeta}_n}{|\zeta - z|^{2n}}$$

$$= \lim_{z\to\omega^+} \frac{1}{n} \int_{\partial D} \frac{\partial f}{\partial \zeta_k}(\zeta) K(\zeta, z) + (-1)^{n-1} \frac{1}{n} C_n \lim_{z\to\omega^+} \int_{\partial D} \frac{1}{|\zeta - z|^{2n}} \cdot \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}(\overline{\zeta}_j - \overline{z}_j) d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_n.$$

Owing to [8],

$$\lim_{z\to\omega^+} (-1)^{n-1} C_n \int_{\partial D} \frac{1}{|\zeta - z|^{2n}} \sum_{j=1}^n \frac{\partial f}{\partial \overline{\zeta}_j}(\overline{\zeta}_j - \overline{\zeta}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_n$$

$$= (-1)^{n-1} C_n PV \int_{\partial D} \frac{1}{|\zeta - \omega|^{2n}} \sum_{j=1}^n \frac{\partial f}{\partial \overline{\zeta}_j}(\overline{\zeta}_j - \overline{\omega}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n + \frac{1}{2} A_j \frac{\partial f}{\partial \omega_j}(\omega),$$

where

$$A_j = (-1)^n, j = 1; A_j = 0, j = 2, \cdots, n.$$

Then by Definition 1 we have

$$F_i(\omega) = \lim_{z\to\omega^+} \frac{1}{n}[PV \int_{\partial D} \frac{\partial f}{\partial \zeta_k}(\zeta) K(\zeta, \omega) + (-1)^{n-1} C_n PV \int_{\partial D} \frac{1}{|\zeta - z|^{2n}} \sum_{j=1}^n \frac{\partial f}{\partial \overline{\zeta}_j}(\overline{\zeta}_j - \overline{\omega}_j)(-1)^{k-1} d\zeta_1 \wedge \cdots \wedge [d\zeta_k] \wedge \cdots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_n] + \frac{1}{2n}[\frac{\partial f}{\partial \omega_k}(\omega) + A_j \frac{\partial f}{\partial \omega_j}(\omega)]$$

$$= FP \int_{\partial D} f(\zeta) \frac{\overline{\zeta}_k - \overline{\omega}_k}{|\zeta - \omega|^2} K(\zeta, \omega) + \frac{1}{2n}\left[\frac{\partial f}{\partial \omega_k}(\omega) + (-1)^n \frac{\partial f}{\partial \omega_1}(\omega)\right],$$

which is (10). The equality (11) is proved similarly.

3 Composite Formula

Lemma 2 (Composite formula for the Cauchy singular integral) Suppose that $\phi(\zeta) \in C^{(1)}(\partial D)$, and that it can be holomorphically extended into D, then

$$\int_{\partial D_\eta} K(\zeta, \eta) \int_{\partial D_\xi} \phi(\xi) K(\eta, \xi) = \frac{1}{4}\phi(\zeta). \tag{12}$$

For this Lemma we refer the reader to [3,4,10].

Theorem 2 (Composite formula for higher order singular integrals) Suppose that $\phi(\xi)$ is holomorphic in a neighbourhood of ∂D, that is $\dfrac{\partial\phi(\xi)}{\partial\overline{\xi}}=0$ in $U(\partial D)$, then the composite formula

$$\int_{\partial D_\eta} K(\zeta,\eta)\int_{\partial D_\xi}\phi(\xi)\frac{\overline{\eta}_k-\overline{\zeta}_k}{|\eta-\xi|^2}K(\eta,\xi)=\frac{1}{4}\frac{\partial\phi(\zeta)}{\partial\zeta_k} \tag{13}$$

holds. In the notation

$$S\phi=2\int_{\partial D_\xi}\phi(\xi)K(\eta,\xi) \tag{14}$$

$$S_1\phi=2\int_{\partial D_{xi}}\phi(\xi)K_1(\eta,\xi),\qquad K_1(\eta,\xi)=\frac{\overline{\eta}_k-\overline{\xi}_k}{|\eta-\xi|^2}K(\eta,\xi), \tag{15}$$

the composite formula can be written as

$$SS_1\phi=4\int_{\partial D_\eta}K(\zeta,\eta)\int_{\partial D_\xi}\phi(\xi)K_1(\eta,\xi)=\frac{1}{n}\frac{\partial\phi(\zeta)}{\partial\zeta_k}. \tag{16}$$

Proof The assumption $\dfrac{\partial\phi(\zeta)}{\partial\overline{\zeta}}=0$ in $U(\partial D)$ implies that $\phi(\zeta)$ can be holomorphically extended into D. Hence the composite formula (12) for $\dfrac{\partial\phi(\zeta)}{\partial\zeta_k}$ holds. Owing to Definition 1 we have

$$\begin{aligned}
&\int_{\partial D_\eta}K(\zeta,\eta)\int_{\partial D_\xi}\phi(\xi)\frac{\overline{\eta}_k-\overline{\xi}_k}{|\eta-\xi|^2}K(\eta,\xi)\\
=&\int_{\partial D_\eta}K(\zeta,\eta)\int_{\partial D_\xi}\frac{1}{n}C_n d\phi(\xi)\cdot\\
&\cdot\frac{\sum_{j=1}^n(-1)^{j-1}(-1)^{k-1}(\overline{\xi}_j-\overline{\eta}_j)d\xi_1\wedge\cdots\wedge[d\xi_k]\wedge\cdots\wedge d\xi_n d\overline{\xi}_1\wedge\cdots\wedge[d\overline{\xi}_k]\wedge\cdots\wedge d\overline{\xi}_n}{|\eta-\xi|^{2n}}\\
=&\frac{1}{n}\int_{\partial D_\eta}K(\zeta,\eta)\int_{\partial D_\xi}C_n\frac{\partial\phi(\xi)}{\partial\xi_k}\frac{\sum_{j=1}^n(-1)^{j-1}(\overline{\xi}_j-\overline{\eta}_j)d_{\xi_1}\wedge\cdots\wedge d\xi_n\wedge d\overline{\xi}_1\wedge\cdots\wedge[d\overline{\xi}_k]\wedge\cdots\wedge d\overline{\xi}_n}{|\eta-\xi|^{2n}}\\
=&\frac{1}{n}\int_{\partial D_\eta}K(\zeta,\eta)\int_{\partial D_\xi}\frac{\partial\phi(\xi)}{\partial\xi_k}K(\eta,\xi)=\frac{1}{4n}\frac{\partial\phi(\zeta)}{\partial\zeta_k}.
\end{aligned}$$

The proof is complete.

Remark Assume that D is a complex ball and $\phi=P_{s,t}$ is a homogeneous harmonic polynomial of degree s with respect to z, and degree t with respect to $\overline{z}$. Then for $n>1$, the composite formula (12) in the case of the Cauchy singular integral cannot hold, unless when $P_{s,t}$ is a holomorphic polynomial, i.e. when $t=0$. So in Lemma 2 we assume that $\phi(\zeta)\in C^{(1)}(\partial D)$, and that it can be holomorphically extended into D. Likewise, in Theorem 2 we assume that $\phi(\xi)$ is holomorphic in a neighbourhood of ∂D. It is a sufficient condition for the composite formula (13) to hold.

Recall that

$$S_1\phi=2\int_{\partial D_\xi}\phi(\xi)K_1(\eta,\xi)=\psi(\eta),\qquad K_1(\eta,\xi)=\frac{\overline{\eta}_k-\overline{\xi}_k}{|\eta-\xi|^2}K(\eta,\xi). \tag{17}$$

Applying the operator

$$S\phi = 2\int_{\partial D_\xi} \phi(\xi)K(\eta,\xi)$$

to both sides of (17), we have

$$SS_1\phi = \frac{1}{n}\frac{\partial\phi(\zeta)}{\partial\zeta_k} = S\psi = \int_{\partial D_\eta} \psi(\eta)K(\zeta,\eta), \tag{18}$$

Under suitable boundary conditions we can uniquely solve the above partial differential equation for ϕ.

4 Higher Order Singular Integral Equations and Partial Differential Integral Equations

In this section we discuss the linear space L which consists of complex value differentiable functions whose partial derivatives satisfy the Hölder condition on ∂D. In fact, the higher order Bochner-Martinelli singular integrals induce linear operators on L. In the following we solve the higher order singular integral equations with Bochner-Martinelli kernel.

We consider the higher order singular integral equation

$$aS\phi + bS_1\phi + T\phi = \psi, \tag{19}$$

where a, b are complex constants, $\psi \in L$ is a given function, $T\phi = \int_{\partial D_\xi} \phi(\xi)L(\eta,\xi)$, where the kernel $L(\eta,\xi)$ is a complex exterior differential form of degree $2n-1$ and $L(\eta,\xi) \in H_1$, i.e. the first order partial devivatives of $L(\eta,\xi)$ with respect to η and ξ satisfy the Hölder condition on ∂D.

Firstly, we consider the characteristic equation of (19)

$$aS\phi + bS_1\phi = \psi. \tag{20}$$

Indeed, by Definition 1, it is a differential and singular integral equation.

Applying the operator

$$M = aI - bS, \tag{21}$$

where

$$I\phi = \phi \qquad (\phi \in L), \tag{22}$$

to both sides of (20), and utilizing the composite formula (16), we have

$$a\phi + \frac{b}{n}\frac{\partial\phi(\zeta)}{\partial\zeta_k} = S\psi. \tag{23}$$

Solving this partial differential equation under suitable boundary value conditions we can obtain the unknown function ϕ uniquely.

The general equation (19) can be reduced to a equivalent Fredholm equation by using the operator M. In fact, applying the operator M to equation (19) and using composite formula (16), we have

$$a\phi + \frac{b}{n}\frac{\partial\phi(\zeta)}{\partial\zeta_k} + ST\phi = S\psi. \tag{24}$$

Applying the theorem of exchanging order of integration for singular integral and ordinary integral ([2] Th.4.5.1), we have

$$\begin{aligned} ST\phi &= \int_{\partial D_\eta} K(\zeta,\eta) \int_{\partial D_\xi} \phi(\xi) L(\eta,\xi) \\ &= \int_{\partial D_\xi} \phi(\xi) \int_{\partial D_\eta} L(\eta,\xi) K(\zeta,\eta). \end{aligned}$$

We can then prove that $\int_{\partial D_\eta} L(\eta,\xi)K(\zeta,\eta)$ satisfies the Hölder condition with respect to ξ and ζ ([2] p.292). Therefore, (24) is a Fredholm equation.

Equation (24) and (19) are equivalent. To show this we need only prove that the solution of (24) satisfies equation (19). In fact, applying the operator

$$M^* = aI + bS \tag{25}$$

to both sides of (24) and use composite formula (16) we obtain (19) immediately.

To summarise, we have

Theorem 3 Suppose that in equation (19), a, b are complex constants, $\psi \in L$, and the kernel $L(\eta,\xi)$ of operator T belongs to H_1, then

(i) The solution of the characteristic equation (20) of (19) can be obtained uniquely in L by solving the partial differential equation (23) under suitable boundary value conditions.

(ii) In L, the higher order singular integral equation (19) is equivalent to the Fredholm equation (24).

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