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# Structure of Solutions of Polynomial Dirac Equations in Clifford Analysis

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In this note, structures of null solutions of the polynomial Dirac operators $D-\lambda$, $D^k$, $D^n+\sum_{j=0}^{n-1} b_j D^j$ are studied, where $D$ is the Dirac operator in $\mathbf{R}_1^m$, $\lambda, b_j \in \mathbf{C}, j=0,\ldots,n-1$, $D^0=I$ is the identity operator. Explicit decompositions of null solutions of the polynomial Dirac operators in the respectively relevant subspaces are obtained which are used to derive their Taylor series expansions. The solutions of inhomogeneous equation $p(D)f=g$ are discussed for a special class of $\mathbf{R}^{(m)}$-valued continuous functions $g$.

*Keywords:* Dirac operator; Monogenic function; Polynomial Dirac operator

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## 1 INTRODUCTION

Let $\mathbf{R}^{(m)}$ be the real associative Clifford algebra generated by $\{e_1, e_2, \ldots, e_m\}$, where the basic vectors $e_1, \ldots, e_m$ satisfy the relations $e_ie_j + e_je_i = -2\delta_{ij}, i,j = 1, \ldots, n$. Viewed as a linear algebra $\mathbf{R}^{(m)}$ is real $2^m$-dimensional with the basis elements $e_0, e_1, \ldots, e_m$, $e_1e_2, \ldots, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_m$, $1 \le j_1 < \cdots < j_r \le m$, where $e_0 = 1$ is the algebraic unit element. Similarly, denote by $\mathbf{C}^{(m)}$ the Complex Clifford algebra generated by $\{e_1, e_2, \ldots, e_m\}$.

Denote by $\mathbf{R}^m$ the real linear subspace of $\mathbf{R}^{(m)}$ spanned by $\{e_1, e_2, \ldots, e_m\}$. A typical element of $\mathbf{R}^m$ is denoted by $\underline{x} = x_1e_1 + \cdots + x_me_m, x_j \in \mathbf{R}$. Define $\mathbf{R}_1^m = \{x = x_0 + \underline{x} | \, x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^m\}$. The *Dirac operator in* $\mathbf{R}^m$ is defined to be $\underline{D} = \sum_{j=1}^m e_j(\partial/\partial x_j)$. The *Dirac operator in* $\mathbf{R}_1^m$ is $D = (\partial/\partial x_0) + \underline{D}$. The Dirac operators are generalizations of the Cauchy–Riemann operator in the complex plane. The operators $\underline{D}^2 = -\sum_{j=1}^m \partial^2/\partial x_j^2$, and, with $\overline{D} = (\partial/\partial x_0) - \underline{D}$, $D\overline{D} = \overline{D}D = \sum_{j=0}^m \partial^2/\partial x_j^2$, are respectively the Laplace operators in $\mathbf{R}^m$ and $\mathbf{R}_1^m$.

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Let $\Omega$ be a domain in $\mathbf{R}_1^m$. If $f:\Omega\to\mathbf{R}^{(m)}$ is a $C^1$ function satisfying $(Df)(x)=\sum_{j=0}^{m}e_j(\partial f/\partial_{x_j})=0$, then $f$ is said to be *left-monogenic* in $\Omega$. The set of left-monogenic functions in $\Omega$ forms a right-module, denoted by $M_{(r)}(\Omega;\mathbf{R}^{(m)})$. We only consider left-monogenic functions and we omit the subscript $(r)$ in $M_{(r)}(\Omega;\mathbf{R}^{(m)})$ as discussed below. Monogenic functions in $\mathbf{R}^m$ are defined similarly.

It is known that in a number of aspects monogenic functions are analogous to analytic functions in one complex variable. For instance, both of them have Cauchy–Green formula, Cauchy integral formula, Maximum Modulus Theorem, Morera's Theorem (see [1]) etc. There have been studies on null solutions of the operators $\underline{D}-\lambda$ [2], $D-M$, $M$ any bounded operator commuting with all $e_j(\partial/\partial x_j)$ [3], $\underline{D}-b(x)$ [4] and $D^k,\underline{D}^k$ ([5,6]) etc. In [7] fundamental solutions of polynomial Dirac equations $p(\underline{D})=(\underline{D}^n+\sum_{j=0}^{n-1}b_j\underline{D}^j)$ in $\mathbf{R}^m$ are constructed. In [8] Ryan obtained the Cauchy–Green formula for null solutions of $p(\underline{D})$.

In this article, structures of solutions of $D-\lambda$, $(D-\lambda)^k$, $p(D)=D^n+\sum_{j=0}^{n-1}b_jD^j$ are studied. Decompositions of null solutions of $p(D)$ in the relevant subspaces are obtained with which their Taylor series expansions are deduced. These results reveal that solutions of polynomial Dirac operators in $\mathbf{R}_1^m$ are closely related to monogenic functions and null solutions of ordinary differential equations $(d^n/dx_0^n)+\sum_{j=0}^{n-1}b_j\times(d^j/dx_0^j)=0$. As application, in Section 5 solutions of inhomogeneous equations $p(D)f=g$ are discussed for a special class of $\mathbf{R}^{(m)}$-valued continuous functions $g$.

## 2 THE SOLUTIONS OF $(D-\lambda)f=0$

In the following, assume that $\Omega$ is a domain (open and connected) in $\mathbf{R}_1^m$ containing the origin and denote $\Omega_0=\{x_0\in R\mid\exists\ \underline{x}\text{ such that }(x_0,\underline{x})\in\Omega\}$. We have

LEMMA 1 *Let* $g\in C^1(\Omega,\mathbf{C}^{(m)})$ *and h be a scalar valued differentiable function defined in* $\Omega_0$, *then*

$$D(hg)=(Dh)g+h(Dg)=h'(x_0)g(x)+h(x_0)(Dg)(x).$$

LEMMA 2 *For any* $f\in C^1(\Omega,\mathbf{C}^{(m)})$, $\lambda\in C$, *we have*

$$(D-\lambda)f(x)=e^{\lambda x_0}D(e^{-\lambda x_0}f)(x).$$

The above two Lemmas can be proved through direct computation.

Denote $\ker(D-\lambda)=\{f\mid f\in C^1(\Omega)\text{ such that }(D-\lambda)f=0\}$, where $\lambda\in C$. In below, denote $h(x_0)M(\Omega;\mathbf{R}^{(m)})=\{f\mid f=h(x_0)g(x),g\in M(\Omega;\mathbf{R}^{(m)})\}$.

THEOREM 1 $\ker(D-\lambda)=e^{\lambda x_0}M(\Omega;\mathbf{R}^{(m)})$.

*Proof* This is an immediate consequence of $(D-\lambda)f(x)=e^{\lambda x_0}D(e^{-\lambda x_0}f)(x)$. ∎

COROLLARY 1 *If* $f\in\ker(D-\lambda)$, *then in a neighborhood of the origin in* $\mathbf{R}_1^m$,

$$f(x)=e^{\lambda x_0}\sum_{k=0}^{\infty}\sum_{(l_1,\dots,l_k)}V_{l_1,\dots,l_k}(x)\left.\frac{\partial^k(e^{-\lambda x_0}f)}{\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}}\right|_{x=0},\qquad(1)$$

*where*

$$V_0(x) = e_0, \qquad V_l = z_l, \qquad V_{l_1,\ldots,l_j}(x) = \frac{1}{j!}\sum_{\pi(l_1,\ldots,l_j)} z_{l_1} z_{l_2} \cdots z_{l_j}, \tag{2}$$

$(l_1,\ldots,l_j) \in \{1,\ldots,m\}^j$, *and* $z_l = x_l e_0 - x_0 e_l, l = 1,\ldots,m$, *and* $\pi(l_1,\ldots,l_j)$ *runs over all distinguishable permutations of* $(l_1,\ldots,l_j)$.

*Proof* If $f$ is the solution of $(D-\lambda)f = 0$, then from Theorem 1 $e^{-\lambda x_0} f(x)$ is a monogenic function. Thus by Taylor expansion of monogenic functions in [1], we have

$$e^{-\lambda x_0} f(x) = \sum_{k=0}^{\infty} \sum_{(l_1,\ldots,l_k)} V_{l_1,\ldots,l_k}(x) \frac{\partial^k (e^{-\lambda x_0} f)}{\partial_{x_{l_1}} \cdots \partial_{x_{l_k}}}\bigg|_{x=0}.$$

■

*Remark 1* By a different method Kisil proved similar results in [3] for operator $D - M$, where $M$ is any bounded operator commuting with all $e_j \partial/\partial x_j$. He proved that solutions $f$ of $(D-M)f = 0$ can be represented as $f(x) = e^{Mx_0} g(x)$, where $g$ is a monogenic function. Theorem 1 is a particular case of this general result as the multiplication operators $M_\lambda$ defined by $M_\lambda f = \lambda f$ for complex numbers $\lambda$ commute with all $e_j \partial/\partial x_j$.

*Remark 2* In [4] Ryan proved that if $f(\underline{x})$ is a solution of $(\underline{D} - b(\underline{x}))f = 0$, where $b(\underline{x}) = (\underline{D}B)(\underline{x})$ and $B(\underline{x})$ is a real valued $C^1$ function, then $f(\underline{x})$ is of the form $f(\underline{x}) = e^{B(\underline{x})} f_1(\underline{x})$, where $f_1(\underline{x})$ is a monogenic function with respect to $\underline{D}$, and vice versa. This may be regarded as the analogous result of Theorem 1 in $\mathbf{R}^m$ for $\underline{D}$.

## 3 THE SOLUTIONS OF $D^k f = 0$

There has been literature on function theory of null solutions of $D^k$ and $\underline{D}^k$ (for instance, see [5,6]), where the null solutions are called $(k)$-monogenic functions. In this article we use the notation $\ker(D^k) = \{f \mid D^k f(x) = 0, f \in C^k\}$ and $\ker(\underline{D}^k) = \{f \mid \underline{D}^k f(\underline{x}) = 0\}$.

LEMMA 3 *The functions* $x_0^j h(x)$, $h \in M(\Omega; \mathbf{R}^{(m)})$, $j = 0, 1, \ldots, k-1$, *belong to* $\ker(D^k)$, *and thus* $\cup_{j=0}^{k-1} x_0^j M(\Omega; \mathbf{R}^{(m)}) \subset \ker(D^k)$.

*Proof* Let $h \in \ker(D)$, i.e., $Dh = 0$, by Lemma 1,

$$D^k(x_0^j h) = D^{k-1}(D(x_0^j h)) = D^{k-1}((x_0^j)' h + x_0^j (Dh)) = j D^{k-1}(x_0^{j-1} h).$$

Then by induction, for any $0 \le j \le k-1$,

$$D^k(x_0^j h) = j D^{k-1}(x_0^{j-1} h) = \cdots = j!\, D^{k-j} h = 0.$$

■

THEOREM 2 (see [5]) *If $f$ is left-$(k)$-monogenic in a domain* $\Omega \subset \mathbf{R}_1^m$, *then*

1. *$f$ is real analytic in* $\Omega$;

2. *Suppose that the origin $O \in \Omega$, then there is a neighborhood of $O$ in $\Omega$ in which $f$ can be written as*

$$f(x) = \sum_{n=0}^{\infty} \sum_{j=0}^{\inf(n,k-1)} \sum_{(l_1,\ldots,l_{n-j})} \frac{x_0^j}{j!} V_{l_1,\ldots,l_{n-j}}(x) \frac{\partial^{n-j} D^j f}{\partial x_{l_1} \cdots \partial x_{l_{n-j}}}\bigg|_{x=0}, \tag{3}$$

*where* $(l_1, \ldots, l_{n-j}) \in \{1, \ldots, m\}^{n-j}$.

Since series (3) is absolutely convergent in an open neighborhood of the origin $O$, it can also be written as

$$\begin{aligned}
f(x) &= \left(\sum_{n=0}^{k-1}\sum_{j=0}^{n} + \sum_{n=k}^{\infty}\sum_{j=0}^{k-1}\right) \sum_{(l_1,\ldots,l_{n-j})} \frac{x_0^j}{j!} V_{l_1,\ldots,l_{n-j}}(x) \frac{\partial^{n-j} D^j f}{\partial x_{l_1} \cdots \partial x_{l_{n-j}}}\bigg|_{x=0} \\
&= \left(\sum_{j=0}^{k-1}\sum_{n=j}^{k-1} + \sum_{j=0}^{k-1}\sum_{n=k}^{\infty}\right) \sum_{(l_1,\ldots,l_{n-j})} \frac{x_0^j}{j!} V_{l_1,\ldots,l_{n-j}}(x) \frac{\partial^{n-j} D^j f}{\partial x_{l_1} \cdots \partial x_{l_{n-j}}}\bigg|_{x=0} \\
&= \sum_{j=0}^{k-1} \frac{x_0^j}{j!} \left( \sum_{n=j}^{\infty} \sum_{(l_1,\ldots,l_{n-j})} V_{l_1,\ldots,l_{n-j}}(x) \frac{\partial^{n-j} D^j f}{\partial x_{l_1} \cdots \partial x_{l_{n-j}}}\bigg|_{x=0} \right) \\
&= \sum_{j=0}^{k-1} \frac{x_0^j}{j!} \left( \sum_{n=0}^{\infty} \sum_{(l_1,\ldots,l_n)} V_{l_1,\ldots,l_n}(x) \frac{\partial^{n} D^j f}{\partial x_{l_1} \cdots \partial x_{l_n}}\bigg|_{x=0} \right) \\
&= \sum_{j=0}^{k-1} \frac{x_0^j}{j!} f_j(x),
\end{aligned}$$

where $f_j(x),\ j = 0, 1, \ldots, k-1$, are monogenic in the open neighborhood of the origin $O$. We therefore have

THEOREM 3 $\ker(D^k)$ *has the direct sum decomposition*:

$$\ker(D^k) = M(\Omega; \mathbf{R}^{(m)}) \oplus x_0 M(\Omega; \mathbf{R}^{(m)}) \oplus \cdots \oplus x_0^{k-1} M(\Omega; \mathbf{R}^{(m)}). \tag{4}$$

In a recent paper ([9]) Ryan proved that if $\underline{D}^k f(\underline{x}) = 0$, then

$$f(\underline{x}) = f_0(\underline{x}) + \underline{x} f_1(\underline{x}) + \cdots + \underline{x}^{k-1} f_{k-1}(\underline{x}), \tag{5}$$

where $\underline{D}\, f_j(\underline{x}) = 0, j = 0, \ldots, k-1$. This result is considered to be analogous to what we have in Theorem 3. On the other hand, direct computation shows that in the context $\mathbf{R}_1^m$ for a non-zero monogenic function $f$, the function $D^2(xf)$ is no longer identical to zero in general. This shows that the statement made from the above mentioned Ryan's result by replacing $\underline{D}$ by $D$ and $\underline{x}$ by $x$ is not true in $\mathbf{R}_1^m$.

From Lemma 3 $x_0^{k-1} M(\Omega; \mathbf{R}^{(m)}) \subset \ker(D^k)$. So Theorem 3 can also written as

$$\ker(D^k) = \ker(D^{k-1}) \oplus x_0^{k-1} M(\Omega; \mathbf{R}^{(m)}). \tag{6}$$

Note that for $f \in \ker(D^k)$ the reduction in the proof of Lemma 3 shows that $g = f - 1/(k-1)!\, x_0^{k-1} D^{k-1} f \in \ker(D^{k-1})$. Hence $f$ has an alternative decomposition $f(x) = g(x) + 1/(k-1)!\, x_0^{k-1} D^{k-1} f$, where $g(x)$ is a $(k-1)$-monogenic function.

## 4 THE SOLUTIONS OF $p(D)f = 0$

For a polynomial $p(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_n$, $b_j \in \mathbf{C}$, $j = 1, \ldots, n$, one can associate it with a polynomial Dirac operators $p(D) = D^n + b_1 D^{n-1} + \cdots + b_n$. Polynomial $p(\lambda)$ is called the characteristic polynomial of $p(D)$. In [7,8] fundamental solutions of $p(\underline{D})$ in $\mathbf{R}^m$ and function theory of solutions of $p(\underline{D})$ are studied.

Denote $\ker(p(D)) = \{f \mid p(D)f = 0, f \in C^n(\Omega; \mathbf{C}^{(m)})\}$. The set $\ker(p(D))$ is a right Hilbert $\mathbf{C}^{(m)}$-module. Since $p(\lambda)$ has the decomposition

$$p(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n), \tag{7}$$

where $\lambda_i \in \mathbf{C}$, $i = 1, \ldots, n$, are the solutions of $p(\lambda) = 0$, the associated polynomial Dirac operator $p(D)$ has the decomposition

$$p(D) = (D - \lambda_1) \cdots (D - \lambda_n). \tag{8}$$

Every operator $D - \lambda_i$ in (8) commutes with the others.

LEMMA 4 *Let* $\pi(\lambda) = \prod_{k=1}^{l} (\lambda - \lambda_k)^{n_k}$ *be a polynomial of* $\lambda$, $n_k \in N$, *then*

$$\frac{1}{\pi(\lambda)} = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{1}{(n_k - j)!} \left[ \frac{d^{n_k - j}}{d\lambda^{n_k - j}} \frac{(\lambda - \lambda_k)^{n_k}}{\pi(\lambda)} \right]_{\lambda = \lambda_k} \frac{1}{(\lambda - \lambda_k)^j}. \tag{9}$$

*Proof* The rational function $1/\pi(\lambda)$ has a decomposition into partial fractions:

$$\frac{1}{\pi(\lambda)} = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{a_{k,j}}{(\lambda - \lambda_k)^j}.$$

We have, for $1 \leq s \leq l$,

$$\frac{(\lambda - \lambda_s)^{n_s}}{\pi(\lambda)} = \sum_{k=1, k \neq s}^{l} \sum_{j=1}^{n_k} \frac{a_{k,j}(\lambda - \lambda_s)^{n_s}}{(\lambda - \lambda_k)^j} + \sum_{j=1}^{n_s} a_{s,j}(\lambda - \lambda_s)^{n_s - j}.$$

Clearly,

$$\left[ \frac{d^{n_s - j}}{d\lambda^{n_s - j}} \frac{(\lambda - \lambda_s)^{n_s}}{\pi(\lambda)} \right]_{\lambda = \lambda_s} = (n_s - j)!\, a_{s,j}, \quad j = 1, \ldots, n_s.$$

■

In below we first study the case that all $\lambda_j$ in the decomposition (8) are different. In this case the coefficients in (9) can be easily worked out.

THEOREM 4 *If $\lambda_j$ in* (8), $j = 1, \ldots, n$, *are all different, then*

$$\ker(p(D)) = \ker(D - \lambda_1) \oplus \cdots \oplus \ker(D - \lambda_n). \tag{10}$$

*Proof* In the assumed case we have $l = n$ and $l_k = 1$ in Lemma 4. Through simple computation (or refer to Lagrange's formula of interpolation) we have the identical relation

$$1 = \sum_{k=1}^{n} \prod_{j \neq k}^{n} \frac{\lambda - \lambda_j}{\lambda_k - \lambda_j}$$

which implies the identical relation for the Dirac operator $D$, viz.

$$I = \sum_{k=1}^{n} \prod_{j \neq k}^{n} \frac{D - \lambda_j}{\lambda_k - \lambda_j},$$

where $I$ is the identity operator. So, for any $f \in C^{n-1}(\Omega; \mathbf{C}^m)$,

$$f = \sum_{k=1}^{n} \left( \prod_{j \neq k}^{n} \frac{D - \lambda_j}{\lambda_k - \lambda_j} \right) f. \tag{11}$$

Since $f \in \ker(p(D))$, we have

$$\left( \prod_{j \neq k}^{n} \frac{D - \lambda_j}{\lambda_k - \lambda_j} \right) f \in \ker(D - \lambda_k).$$

■

COROLLARY 2 *If $\lambda_j$ in* (8), $j = 1, \ldots, n$, *are all different, then*

$$\ker(p(D)) = e^{\lambda_1 x_0} M(\Omega; \mathbf{R}^{(m)}) \oplus \cdots \oplus e^{\lambda_n x_0} M(\Omega; \mathbf{R}^{(m)}). \tag{12}$$

COROLLARY 3 *If $\lambda_j$ in* (8), $j = 1, \ldots, n$, *are all different, and $f \in \ker(p(D))$, then the Taylor expansions of $f$ is*

$$f(x) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} \sum_{(l_1, \ldots, l_j)} \frac{e^{\lambda_k x_0} V_{l_1, \ldots, l_j}(x)}{\prod_{i \neq k}^{n} (\lambda_k - \lambda_i)} \frac{\partial^j (e^{-\lambda_k x_0} (\prod_{i \neq k}^{n} (D - \lambda_i)) f)}{\partial_{x_{l_1}} \cdots \partial_{x_{l_j}}} \Bigg|_{x=0}.$$

*Proof* This is an immediate consequence of (11) and (1). ■

THEOREM 5 *If $p(D) = (D - \lambda)^n$ in* (8), *then* $\ker((D - \lambda)^n) = e^{\lambda x_0} \ker(D^n)$ *and*

$$\ker((D - \lambda)^n) = e^{\lambda x_0} M(\Omega; \mathbf{R}^{(m)}) \oplus x_0 e^{\lambda x_0} M(\Omega; \mathbf{R}^{(m)}) \oplus \cdots \oplus x_0^{n-1} e^{\lambda x_0} M(\Omega; \mathbf{R}^{(m)}). \tag{13}$$

*Proof* Inductively, we have

$$(D - \lambda)^n f = (D - \lambda)^{n-1} (e^{\lambda x_0} D(e^{-\lambda x_0} f)(x)) = \cdots = e^{\lambda x_0} D^n (e^{-\lambda x_0} f)(x).$$

The result then follows from Theorem 3. ■

COROLLARY 4 *If* $f \in \ker((D-\lambda)^n)$, *then* $f$ *has the Taylor expansions at the origin* $O$

$$f(x) = e^{\lambda x_0} \sum_{k=0}^{\infty} \sum_{j=0}^{\inf(n-1,k)} \sum_{(l_1,\ldots,l_{k-j})} \frac{x_0^j}{j!} V_{l_1,\ldots,l_{k-j}}(x) \left. \frac{\partial^{k-j} D^j (e^{-\lambda x_0} f)}{\partial x_{l_1} \cdots \partial x_{l_{k-j}}} \right|_{x=0}.$$

*Proof* Theorem 5 asserts that $f \in \ker((D-\lambda)^n)$ implies $e^{-\lambda x_0} f \in \ker(D^n)$. The assertion then follows from Theorem 2.

Now we study the general case. Let $\lambda_1,\ldots,\lambda_l$ be all the different roots in (8) with corresponding multiples $n_1,\ldots,n_l$, $n_1+\cdots+n_l=n$, $n_j \in N$, $j=1,\ldots,l$. The polynomial Dirac operator $p(D)$ in (8) can be written as

$$p(D) = (D-\lambda_1)^{n_1} \cdots (D-\lambda_l)^{n_l}. \tag{14}$$

■

THEOREM 6 *If* $p(D)$ *in Eq.* (8) *has the decomposition Eq.* (14), *then*

$$\ker(p(D)) = \ker(D-\lambda_1)^{n_1} \oplus \cdots \oplus \ker(D-\lambda_l)^{n_l}. \tag{15}$$

*Proof* First we note that for any $j$, the operator $(D-\lambda_j)^{n_j}$ commutes with the other $(D-\lambda_i)^{n_i}, i \neq j$. This implies that functions in $\ker((D-\lambda_j)^{n_j})$ belong to $\ker(p(D))$.

On the other hand, let $\pi(\lambda) = \prod_{k=1}^{l} (\lambda-\lambda_k)^{n_k}$ be the characteristic polynomial of $p(D)$. Then by the identical relation (9) in Lemma 4,

$$1 = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{1}{(n_k-j)!} \left[ \frac{d^{n_k-j}}{d\lambda^{n_k-j}} \frac{(\lambda-\lambda_k)^{n_k}}{\pi(\lambda)} \right]_{\lambda=\lambda_k} \frac{\pi(\lambda)}{(\lambda-\lambda_k)^j}. \tag{16}$$

Denote $\pi_{k,j}(\lambda) = \pi(\lambda)/(\lambda-\lambda_k)^j$, $1 \leq j \leq n_k$, we have,

$$\pi_{k,j}(D) = (D-\lambda_1)^{n_1} \cdots (D-\lambda_{k-1})^{n_{k-1}} (D-\lambda_k)^{n_k-j} (D-\lambda_{k+1})^{n_{k+1}} \cdots (D-\lambda_l)^{n_l}. \tag{17}$$

For any $f \in C^{n-1}(\Omega; \mathbf{C}^{(m)})$, we have the decomposition

$$f = \sum_{k=1}^{l} \sum_{j=1}^{n_k} \frac{1}{(n_k-j)!} \left[ \frac{d^{n_k-j}}{d\lambda^{n_k-j}} \frac{(\lambda-\lambda_k)^{n_k}}{\pi(\lambda)} \right]_{\lambda=\lambda_k} \pi_{k,j}(D) f. \tag{18}$$

If $f \in \ker(p(D))$, by (17) we have $\pi_{k,j}(D) f \in \ker((D-\lambda_k)^j)$, $j=1,\ldots,n_k$. Taking into account

$$\ker(D-\lambda_k) \subset \ker((D-\lambda_k)^2) \subset \cdots \subset \ker((D-\lambda_k)^{n_k}), \tag{19}$$

for any $f \in \ker(p(D))$, we have

$$\sum_{j=1}^{n_k} \pi_{k,j}(D) f \in \ker((D-\lambda_k)^{n_k}).$$

The proof is complete. ■

*Remark 3* The proof of Theorems 4 and 6 gives rise to explicit decomposition formulas for functions in $\ker(p(D))$. The decomposition in the proof of Theorem 6 is, in fact, finer than what is stated in Theorem 6. Based on Lagrange's formula of interpolation we have a recursive method to work out all the coefficients in decomposition formulas. As example, if $f \in \ker((D-\lambda_1)^2(D-\lambda_2)^2)$, then replace the identical relation $f = ((D-\lambda_1)/(\lambda_2-\lambda_1))f + ((D-\lambda_2)/(\lambda_1-\lambda_2))f$ into itself, we have

$$\begin{aligned} f &= \frac{D-\lambda_1}{\lambda_2-\lambda_1}\left(\frac{D-\lambda_1}{\lambda_2-\lambda_1}f + \frac{D-\lambda_2}{\lambda_1-\lambda_2}f\right) + \frac{D-\lambda_2}{\lambda_1-\lambda_2}\left(\frac{D-\lambda_1}{\lambda_2-\lambda_1}f + \frac{D-\lambda_2}{\lambda_1-\lambda_2}f\right) \\ &= \frac{(D-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}f + 2\frac{(D-\lambda_1)(D-\lambda_2)}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_2)}f + \frac{(D-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}f \\ &= \frac{(D-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}f + 2\frac{(D-\lambda_1)(D-\lambda_2)}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_2)}\left(\frac{D-\lambda_1}{\lambda_2-\lambda_1}f + \frac{D-\lambda_2}{\lambda_1-\lambda_2}f\right) + \frac{(D-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}f \\ &= \frac{(D-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}f + 2\frac{(D-\lambda_1)^2(D-\lambda_2)}{(\lambda_2-\lambda_1)^2(\lambda_1-\lambda_2)}f + 2\frac{(D-\lambda_1)(D-\lambda_2)^2}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_2)^2}f + \frac{(D-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}f. \end{aligned}$$

Note that in the recursion procedure, an allowed recursion step is one after that there is no power of $(D-\lambda_j)$ exceeding the maximal power $(D-\lambda_j)^{n_j}$, and the recursion steps end when in each term of the summation there is only one $j$ for which the corresponding operator $(D-\lambda_j)$ is with a less power than $(D-\lambda_j)^{n_j}$ but all the others have the maximal powers. This recursion method is clearly applicable to general polynomial operators $p(D) = (D-\lambda_1)^{n_1}\cdots(D-\lambda_l)^{n_l}$.

COROLLARY 5 *Let* $p(D) = (D-\lambda_1)^{n_1}\cdots(D-\lambda_l)^{n_l}$, $n_1+\cdots+n_l = n$, $n_j \in N$, $j = 1,\ldots,l$, *then*

$$\begin{aligned} \ker(p(D)) = {} & e^{\lambda_1 x_0}M(\Omega;\mathbf{R}^{(m)}) \oplus e^{\lambda_1 x_0}x_0 M(\Omega;\mathbf{R}^{(m)}) \oplus \cdots \oplus e^{\lambda_1 x_0}x_0^{n_1-1}M(\Omega;\mathbf{R}^{(m)}) \\ & \oplus e^{\lambda_2 x_0}M(\Omega;\mathbf{R}^{(m)}) \oplus e^{\lambda_2 x_0}x_0 M(\Omega;\mathbf{R}^{(m)}) \oplus \cdots \oplus e^{\lambda_1 x_0}x_0^{n_2-1}M(\Omega;\mathbf{R}^{(m)}) \\ & \oplus \cdots\cdots \\ & \times e^{\lambda_l x_0}M(\Omega;\mathbf{R}^{(m)}) \oplus e^{\lambda_l x_0}x_0 M(\Omega;\mathbf{R}^{(m)}) \oplus \cdots \oplus e^{\lambda_l x_0}x_0^{n_l-1}M(\Omega;\mathbf{R}^{(m)}). \end{aligned}$$

*Proof* It is concluded from Theorems 3, 5 and 6. ■

As a direct application of Theorem 6, the Taylor expansion of $f \in \ker(p(D))$ can also be derived from Corollary 4 and (18).

*Remark 4* It is seen from Corollary 5 that the solutions of $p(D)f = 0$ are closely related to monogenic functions and the linear independent solutions $x_0^k e^{\lambda_j x_0}$, $j = 1,\ldots,l$, $k = 0,\ldots,n_j-1$, of the ordinary differential equation $p(d/dx_0)g(x_0) = ((d^n/dx_0^n) + \sum_{j=0}^{n-1} b_j(d^j/dx_0^j))g(x_0) = 0$, where $p(d/dx_0) = \prod_{k=1}^{l}((d/dx_0)-\lambda_k)^{n_k}$.

*Remark 5* Theorems 4 and 6 still hold for polynomial Dirac operators $p(\underline{D})$ in $\mathbf{R}^m$, while Corollaries 2–5 and Theorem 5, do not remain in the same form for $\ker(p(\underline{D}))$.

## 5 APPLICATION TO THE SOLUTIONS OF $p(D)f = g$

Structures of solutions of polynomial Dirac equations $p(D)f = 0$ have been studied in the previous sections. In this section, solutions of inhomogeneous equations

$$p(D)f = g \tag{20}$$

will be discussed. The following two theorems can be easily proved.

THEOREM 7 *If in the equation $p(D)f = g$ the function $g$ can be decomposed into $g = g_1 + g_2$, and $f_j(x)$ is a solution of $p(D)f = g_j$, $j = 1, 2$, then $f_1 + f_2$ is a solution of $p(D)f = g$.*

THEOREM 8 *Let $f_1(x)$ be a solution of $p(D)f = g$. Then all solutions of $p(D)f = g$ have the form $f(x) = f_1(x) + h(x)$, where $h \in \ker(p(D))$.*

According to Theorems 7 and 8, to solve an equation (20) is reduced to find a particular solution of the equation. There is no general approach for an arbitrary $\mathbf{R}^{(m)}$-valued continuous function $g$. We claim that for functions $g$ of the form $g(x) = H(x_0)G(x)$, where $H$ is a function in a real variable and $G \in M(\Omega; \mathbf{R}^{(m)})$, and therefore any linear combination of such functions, we are able to deduce a particular solution.

Suggested by the theory of linear ordinary differential equations we assume that a particular solution of the Eq. (20) in the case $g(x) = H(x_0)G(x)$, $G \in M(\Omega; \mathbf{R}^{(m)})$, is of the form $f(x) = F(x_0)G(x)$, where $F$ is a function in a real variable. Then, based on the Lemma 1,

$$D^j(F(x_0)G(x)) = \frac{d^j F}{dx_0^j} G(x), \quad j = 1, \ldots, n. \tag{21}$$

Inserting (21) into (20), we are reduced to

$$p\left(\frac{d}{dx_0}\right) F(x_0) = H(x_0). \tag{22}$$

Thus a particular solution $f$ of Eq. (20) with $g(x) = H(x_0)G(x)$, $G \in M(\Omega; \mathbf{R}^{(m)})$, is reduced to a particular solution $F(x_0)$ of the associated ordinary differential equation (22), that can be solved completely for functions like

$$H(x_0) = p_l(x_0)e^{\alpha x_0}, \qquad H(x_0) = e^{\alpha x_0}[p_l^{(1)}(x_0)\cos\beta x_0 + p_l^{(2)}(x_0)\sin\beta x_0],$$

where $p_l(x_0), p_l^{(1)}(x_0), p_l^{(2)}(x_0)$ are polynomials of $x_0$ and $\alpha, \beta$ are real numbers.

Owing to Theorems 7, 8 and the above discussion, for a large class of functions $g$ expressible by a sum of functions of the form $H_j(x_0)G_j(x)$ with $G_j \in M(\Omega; \mathbf{R}^{(m)})$, the Eq. (20) can be solved completely.

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