Course Code: **MATH 2000**
Course Name: **Engineering Mathematics I**

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

<u># Basic Notations/Definitions/Theorems</u>

Let R be the set of all real numbers. Sometimes, we write $R = (-\infty, \infty)$. Let $a, b \in R$ with $a < b$.
I is **<u>a non-empty interval in R</u>** if I is one of the following forms:
(a, b), $(a, b]$, $[a, b)$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$ and R.

Let $R^n = \{(x_1, x_2, \cdots, x_n): x_1, x_2, \cdots, x_n \in R\}$, that is, it is the set of all $n-$ coordinate points.

Each $n-$tuple $(x_1, x_2, \cdots, x_n)$ can be considered as a position vector from the origin $O(0,0, \cdots ,0)$ to the point $P(x_1, x_2, \cdots, x_n)$, that is $\overrightarrow{OP}$.
The norm/length/magnitude of the vector $\overrightarrow{OP}$ is $\|(x_1, x_2, \cdots, x_n)\| = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2}$.

Zero Vector is denoted as $\vec{0}$ (no directions and no magnitudes).

The position vector $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. Sometimes, we denotes it as $\vec{v}$ or $\boldsymbol{v}$.

Sometimes, we write the norm of $\vec{v}$ as $|\vec{v}|$ or $\|\vec{v}\|$.

Notes:
Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, \lambda \in R$.

1. $(a_1, a_2, \cdots, a_n) = (b_1, b_2, \cdots, b_n) \Leftrightarrow a_i = b_i$ for all $i = 1,2, \cdots, n$
2. $(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)$
3. $(a_1, a_2, \cdots, a_n) - (b_1, b_2, \cdots, b_n) = (a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n)$
4. $\lambda(a_1, a_2, \cdots, a_n) = (\lambda a_1, \lambda a_2, \cdots, \lambda a_n)$

<u>Unit vector</u> is a vector with magnitude 1.
Unit vector in the direction of a non-zero vector $\vec{u}$ is $\frac{1}{|\vec{u}|}\vec{u}$.

Let $\vec{u} = (u_1, u_2, \cdots, u_n)$ and $\vec{v} = (v_1, v_2, \cdots, v_n)$.
We define the dot product/inner product $\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$.
Let θ be the angle between the vectors $\vec{u}$ and $\vec{v}$.
We can show that $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot cos\theta$.

<u>Theorem:</u>
Suppose $\vec{u}$ and $\vec{v}$ are non-zero vectors.
$\vec{u}$ and $\vec{v}$ are perpendicular to each other $\Leftrightarrow \vec{u} \cdot \vec{v} = 0$

<u>For three-dimensional case:</u>
We let $\vec{\imath} = (1,0,0)$, $\vec{\jmath} = (0,1,0)$ and $\vec{k} = (0,0,1)$.
For any vector $\vec{u} = (u_1, u_2, u_3)$, we can write $\vec{u} = u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k}$.
Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$.
We define the cross product
$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{\imath} + (u_3v_1 - u_1v_3)\vec{\jmath} + (u_1v_2 - u_2v_1)\vec{k}$.
We can remember this as $\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$.

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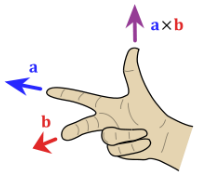
Theorems:
Suppose $\vec{u}$ and $\vec{v}$ are non-zero vectors.
Let θ be the angle between the vectors $\vec{u}$ and $\vec{v}$.

(i) The vectors $\vec{u}$, $\vec{v}$ and $\vec{u} \times \vec{v}$ form a right-handed triple.



(ii) $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot sin\theta$

(iii) $\vec{u}$ and $\vec{v}$ are parallel to each other $\Leftrightarrow \vec{u} \times \vec{v} = \vec{0}$

 # Function of Several Variables
Function of Two Variables:
Let D be a non-empty subset of R^2.
f is called a real-valued function defined on D if for every $(x, y) \in D$, we assign it to exactly one real number.
In this case, we write it as $f(x, y)$. We call $f: D \to R$ a real-valued function and D the domain.

Example:
Let $f: R^2 \to R$ be defined by $f(x, y) = x + y$. f is a real-valued function on R^2.

Function of Three Variables:
Let D be a non-empty subset of R^3. f is called a real-valued function defined on D if for every $(x, y, z) \in D$, we assign it to exactly one real number. In this case, we write it as $f(x, y, z)$.
We call $f: D \to R$ a real-valued function and D the domain.

Example:
Let $f: R^3 \to R$ be defined by $f(x, y, z) = x + y - z$. f is a real-valued function on R^3.

Function of n − Variables:
Let D be a non-empty subset of R^n. f is called a real-valued function defined on D if for every $(x_1, x_2, \cdots, x_n) \in D$, we assign it to exactly one real number. In this case, we write it as $f(x_1, x_2, \cdots, x_n)$.
We call $f: D \to R$ a real-valued function and D the domain.

Example:
Let $f: R^n \to R$ be defined by $f(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n$. f is a real-valued function on R^n.

 # (Natural) Domain of Function of Several Variables
Example 1:
Find the (natural) domains of the functions:

(i) $f(x, y) = \sqrt{25 - x^2 - y^2}$

(ii) $g(x, y, z) = \dfrac{x + y + z}{\sqrt{x^2 + y^2 + z^2}}$

Solutions
The (natural) domains are:

(i) $D = \{(x, y) \in R^2 : x^2 + y^2 \leq 25\}$

(ii) $D = R^3 \backslash \{(0,0,0)\}$

Example 2:

Find the (natural) domain of the function $f(x,y) = \frac{y}{\sqrt{x-y^2}}$. Find also the points (x,y) at which $f(x,y) = \pm 1$.

Solutions

The domain is $\{(x,y) \in R^2 : x - y^2 > 0\}$.

$f(x,y) = \pm 1 \Leftrightarrow \frac{y}{\sqrt{x-y^2}} = \pm 1 \Leftrightarrow y^2 = x - y^2 \Leftrightarrow x = 2y^2$

(Note: We assumed $x - y^2 > 0$)

The points (x,y) at which $f(x,y) = \pm 1$ are given by

$\{(x,y) \in R^2 \backslash \{(0,0)\} : x = 2y^2\}$.

 # Graphs

Let D be a non-empty subset of R^n.

f is a real-valued function defined on D.

We define the graph of f as the set

$\{(x_1, x_2, \cdots, x_n, y) \in R^{n+1} : (x_1, x_2, \cdots, x_n) \in D, y = f(x_1, x_2, \cdots, x_n)\}$

Example 1:

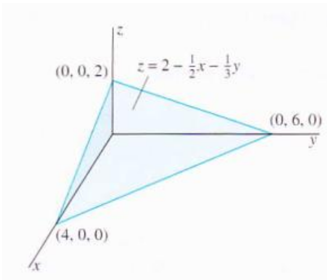
Sketch the graph of the function $f(x,y) = 2 - \frac{1}{2}x - \frac{1}{3}y$.

Solutions

Let $z = f(x,y) = 2 - \frac{1}{2}x - \frac{1}{3}y$ for any $(x,y) \in R^2$.

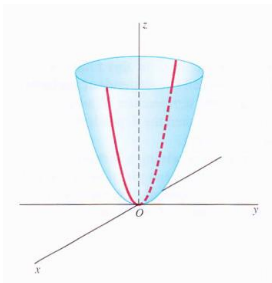
$3x + 2y + 6z = 12$

It is the plane with normal vector $(3,2,6)$ and passing through the point $(0,6,0)$.



Example 2:

The graph of the function $f(x,y) = x^2 + y^2$ is the familiar circular paraboloid $z = x^2 + y^2$ shown in the figure.



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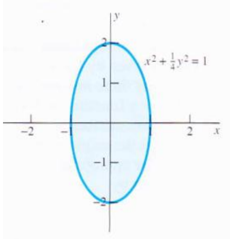
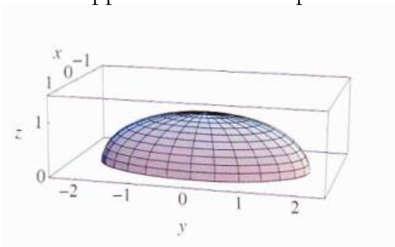
<u>Example 3:</u>
Find the domain of the function $g(x,y) = \frac{1}{2}\sqrt{4-4x^2-y^2}$ and sketch its graph.

<u>Solutions</u>

The domain is
$\{(x,y) \in R^2 : 4x^2 + y^2 \le 4\}$

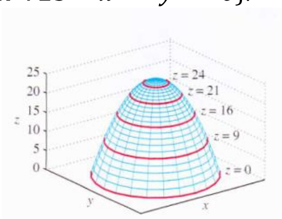


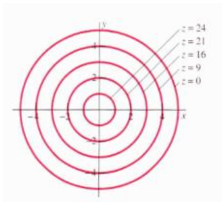
The graph is the upper half of the ellipsoid.



<u># Level Curves/Level Surfaces/Level Sets</u>
Let D be a non-empty subset of R^n. Let $c \in R$. f is a real-valued function defined on D.
We define the level set of f as the set $L_c = \{(x_1, x_2, \cdots, x_n) \in D : f(x_1, x_2, \cdots, x_n) = c\}$
(where the function has the same value c).
When $n = 2$, level set is commonly called level curve.
When $n = 3$, level set is commonly called level surface.

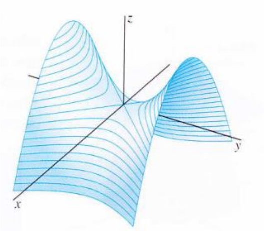
<u>Example 1:</u>
Let $f: R^2 \to R$ be defined by $f(x,y) = 25 - x^2 - y^2$. Domain $= R^2$. Let $c \in R$.
$L_c = \{(x,y) \in R^2 : 25 - x^2 - y^2 = c\}$.

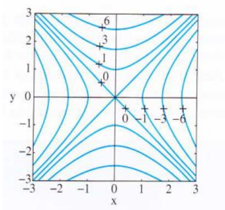




Note: $L_c = \phi$ if $c > 25$ and $L_{25} = \{(0,0)\}$.

<u>Example 2:</u>
Let $f: R^2 \to R$ be defined by $f(x,y) = y^2 - x^2$. Domain $= R^2$. Let $c \in R$.
$L_c = \{(x,y) \in R^2 : y^2 - x^2 = c\}$.

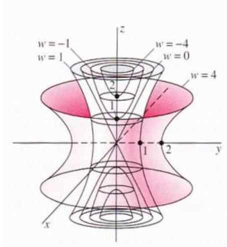




Notes:
(i) If $c > 0$, the level curve $y^2 - x^2 = c$ is a hyperbola opens along the $y-$ axis.
(ii) If $c < 0$, the level curve $y^2 - x^2 = c$ is a hyperbola opens along the $x-$ axis.
(iii) If $c = 0$, the level curve $y^2 - x^2 = 0$ consists of two straight lines given by $y = x$ and $y = -x$.

Example 3:

Let $f: R^3 \to R$ be defined by $f(x, y, z) = x^2 + y^2 - z^2$.
Domain $= R^3$. Let $c \in R$. $L_c = \{(x, y, z) \in R^3: x^2 + y^2 - z^2 = c\}$.

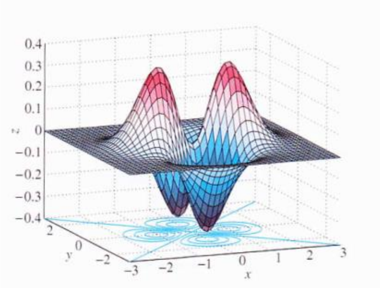


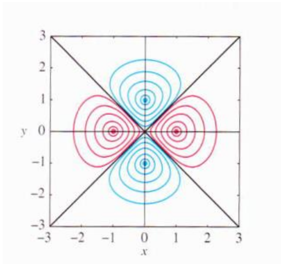
Notes:

(i) If $c > 0$, the level surface $x^2 + y^2 - z^2 = c$ is a hyperboloid of one sheet.
(ii) If $c < 0$, the level surface $x^2 + y^2 - z^2 = c$ is a hyperboloid of two sheets.
(iii) If $c = 0$, the level surface $x^2 + y^2 - z^2 = 0$ is a cone lies between these two types of hyperboloids.

Example 4:

Let $f: R^2 \to R$ be defined by $f(x, y) = (x^2 - y^2)e^{-x^2 - y^2}$. Domain $= R^2$.

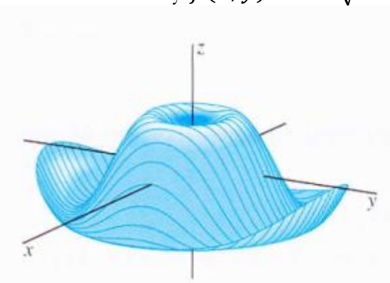


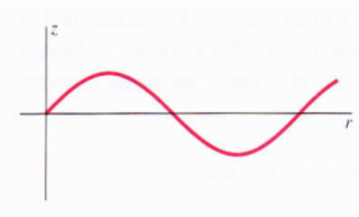


Remark: The patterns of nested level curves can indicate "pits" and "peaks" on the surface.

Example 5:

Let $f: R^2 \to R$ be defined by $f(x, y) = sin\sqrt{x^2 + y^2}$. Domain $= R^2$.





$z = sinr$ where $r = \sqrt{x^2 + y^2}$

Example 6:

Let $f: R^2 \to R$ be defined by $f(x, y) = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 - x^2$. Domain $= R^2$. Investigate the graph of f.

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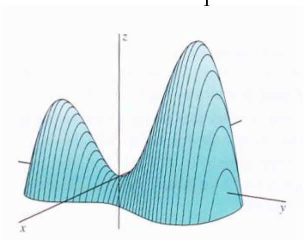
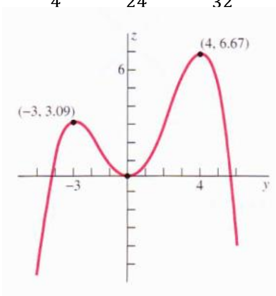
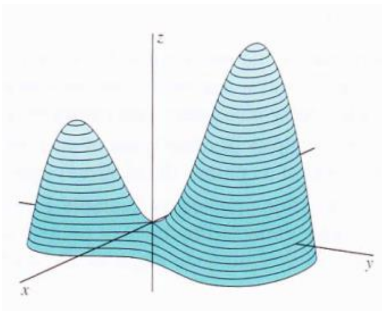
Solutions

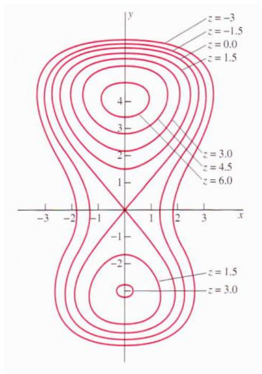
Note 1: If we set $y = y_0$ and let $k = \frac{3}{4}{y_0}^2 + \frac{1}{24}{y_0}^3 - \frac{1}{32}{y_0}^4$, then $f(x, y) = k - x^2$.
$z = k - x^2$ is an equation of a parabola in the $xz -$ plane.



Note 2: If we set $x = 0$, then $f(0, y) = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4$.
$z = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4$ is a curve in the $yz -$ plane.







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Open Sets and Closed Sets in R^n

Definitions:

Let $P(p_1, p_2, \cdots, p_n) \in R^n$ and $r \in R$ with $r > 0$. The open ball centered at P with radius r is $\{(x_1, x_2, \cdots, x_n) \in R^n : \|(x_1, x_2, \cdots, x_n) - (p_1, p_2, \cdots, p_n)\| < r\}$. It is usually denoted as $B(P, r)$.
That is, $B(P, r) = \left\{(x_1, x_2, \cdots, x_n) \in R^n : \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2} < r\right\}$

Let $\phi \neq S \subset R^n$. For any $P \in S$, we can find $r \in R$ with $r > 0$ such that $B(P, r) \subset S$.
S is called **an open set in R^n**.

Let $\phi \neq T \subset R^n$. T is called **a closed set in R^n** if $R^n \backslash T$ is an open set in R^n.

Examples of Open Sets in R^2:

$S_1 = \{(x, y) \in R^2 : x > 1\}$, $S_2 = \{(x, y) \in R^2 : 1 < x^2 + y^2 < 2\}$, $S_3 = R^2 \backslash \{(0,0)\}$

Examples of Closed Sets in R^2:

$T_1 = \{(x, y) \in R^2 : x \leq 1\}$, $T_2 = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\} \cup \{(x, y) \in R^2 : x^2 + y^2 \geq 2\}$, $T_3 = \{(0,0)\}$

Interior Points, Accumulation Points and Boundary Points

Definitions:

Let $\phi \neq S \subset R^n$ and $P \in R^n$.
P is called **an interior point** of S if we can find $r \in R$ with $r > 0$ such that $B(P, r) \subset S$.

P is called **an accumulation point** of S if for any $r \in R$ with $r > 0$, we can find $Q \in R^n$ with $Q \neq P$ and $Q \in B(P, r) \cap S$.

P is called **a boundary point** of S if for any $r \in R$ with $r > 0$, we must have $B(P, r) \cap S \neq \phi$ and $B(P, r) \cap (R^n \backslash S) \neq \phi$.

We define the boundary of S is $\partial S = \{$ all boundary points of S $\}$.

Notes:

(i) P is **an interior point** of $S \Longrightarrow P$ is **an accumulation point** of S

(ii) P is **an accumulation point** of S and is NOT **an interior point** of S

$\Longrightarrow$ P is **a boundary point** of S

(iii) P is **an accumulation point** of $S \Longleftrightarrow$

P is **an interior point** of S or P is **a boundary point** of S

Example:

Let $S = \{(x, y) \in R^2 : x \leq 1\}$.
We can check that:
(i) $(0,0)$ is an interior point of S
(ii) $(1,0)$ is an accumulation point of S and is not an interior point of S
(iii) $(1,0)$ is a boundary point of S

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Bounded Sets and Unbounded Sets in R^n

Definitions:
Let $\phi \neq S \subset R^n$ and $\phi \neq T \subset R^n$
S is **bounded** if we can find $r \in R$ with $r > 0$ such that $S \subset B(O,r)$ where $O(0,0,\cdots,0)$ is the origin.
T is **unbounded** if for any $r \in R$ with $r > 0$, we have $\left(R^n \backslash B(O,r)\right) \cap T \neq \phi$ where $O(0,0,\cdots,0)$ is the origin.

Example:
$S = \{(x,y) \in R^2 : x^2 + y^2 \leq 1\}$ is bounded.
$T = \{(x,y) \in R^2 : x > 1\}$ is unbounded.

Remarks:
Usually we consider:
(i) Limit/Differentiability at accumulation points or on open sets
(ii) Continuity on open sets/closed sets
(iii) Maxima/Minima on closed and bounded sets

Limits and Continuity

For One Dimensional Case:
Recall the definition for $\lim_{x \to a} f(x) = L$:
Let $f: R \to R$ be a function and $a, L \in R$.
For any $\varepsilon > 0$, we can find $\delta > 0$ (δ may depend on ε) such that $0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.

For Two Dimensional Case:
Definition for $\lim_{(x,y) \to (a,b)} f(x,y) = L$:
Let $f: R^2 \to R$ be a function, $(a,b) \in R^2$ and $L \in R$.
For any $\varepsilon > 0$, we can find $\delta > 0$ (δ may depend on ε) such that $0 < \|(x,y) - (a,b)\| < \delta \Longrightarrow |f(x,y) - L| < \varepsilon$.

In this case, we say $f(x,y) \to L$ as $(x,y) \to (a,b)$.
Remark: $\|(x,y) - (a,b)\| = \sqrt{(x-a)^2 + (y-b)^2}$

For $n-$ Dimensional Case:
Definition for $\lim_{(x_1,x_2,\cdots,x_n) \to (a_1,a_2,\cdots,a_n)} f(x_1, x_2, \cdots, x_n) = L$:
Let $f: R^n \to R$ be a function, $(a_1, a_2, \cdots, a_n) \in R^n$ and $L \in R$.
For any $\varepsilon > 0$, we can find $\delta > 0$ (δ may depend on ε) such that
$0 < \|(x_1, x_2, \cdots, x_n) - (a_1, a_2, \cdots, a_n)\| < \delta \Longrightarrow |f(x_1, x_2, \cdots, x_n) - L| < \varepsilon$.
In this case, we say $f(x_1, x_2, \cdots, x_n) \to L$ as $(x_1, x_2, \cdots, x_n) \to (a_1, a_2, \cdots, a_n)$.
Remark: $\|(x_1, x_2, \cdots, x_n) - (a_1, a_2, \cdots, a_n)\| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}$

Uniqueness of Limit
Let $f: R^n \to R$ be a function, $(a_1, a_2, \cdots, a_n) \in R^n$ and $L_1, L_2 \in R$.
If $\lim_{(x_1,x_2,\cdots,x_n) \to (a_1,a_2,\cdots,a_n)} f(x_1, x_2, \cdots, x_n) = L_1$ and $\lim_{(x_1,x_2,\cdots,x_n) \to (a_1,a_2,\cdots,a_n)} f(x_1, x_2, \cdots, x_n) = L_2$, then $L_1 = L_2$.

Proof: Omitted (As Exercise)

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Example 1:

Let $f: R^2 \to R$ be defined by $f(x, y) = xy$ and $(a, b) = (2,3)$. Show that $\lim_{(x,y)\to(a,b)} f(x, y) = 6$.

Proof:

For any $\varepsilon > 0$, choose $\delta = min\left\{1, \frac{\varepsilon}{7}\right\} > 0$

$0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta \Rightarrow 0 < |x-2| < \delta \Leftrightarrow 2 - \delta < x < 2 + \delta$ and $x \neq 2$

$0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta \Rightarrow 0 < |y-3| < \delta \Leftrightarrow 3 - \delta < y < 3 + \delta$ and $y \neq 3$

As $0 < \delta \leq 1$, both $2 - \delta > 0$ and $3 - \delta > 0$.

So, $0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta \Rightarrow (2-\delta)(3-\delta) < xy < (2+\delta)(3+\delta)$

$(2-\delta)(3-\delta) < xy < (2+\delta)(3+\delta) \Leftrightarrow -5\delta + \delta^2 < xy - 6 < 5\delta + \delta^2$

As $0 < \delta \leq 1$, $0 < \delta^2 \leq \delta$. So $5\delta + \delta^2 \leq 6\delta < 7\delta \leq \varepsilon$ and $-5\delta + \delta^2 > -5\delta > -7\delta \geq -\varepsilon$.

combining all results,

$0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta \Rightarrow -\varepsilon < xy - 6 < \varepsilon$

that is, $0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta \Rightarrow |f(x, y) - 6| = |xy - 6| < \varepsilon$

Thus, $\lim_{(x,y)\to(a,b)} f(x, y) = 6$.

Example 2:

Determine whether each of the following limits exists and find the limit if it exists:

(i) $$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

(ii) $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

(iii) $$\lim_{(x,y)\to(0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

Solution (i):

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} \frac{x-y}{x+y} = \lim_{x\to 0} \frac{x-0}{x+0} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} \frac{x-y}{x+y} = \lim_{y\to 0} \frac{0-y}{0+y} = \lim_{y\to 0} \frac{-y}{y} = \lim_{y\to 0} -1 = -1$$

So, $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$ doesn't exist.

Solution (ii):

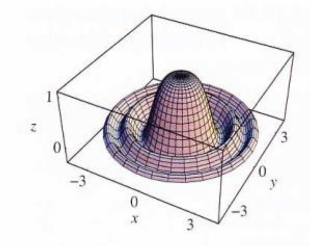
$$\lim_{\substack{(x,y)\to(0,0) \\ y=mx^2}} \frac{x^2 y}{x^4 + y^2} = \lim_{\substack{(x,y)\to(0,0) \\ y=mx^2}} \frac{mx^4}{x^4 + m^2 x^4} = \lim_{\substack{(x,y)\to(0,0) \\ y=mx^2}} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

So, $\lim_{\substack{(x,y)\to(0,0) \\ y=x^2}} \frac{x^2 y}{x^4+y^2} = \frac{1}{2} \neq \frac{-1}{2} = \lim_{\substack{(x,y)\to(0,0) \\ y=-x^2}} \frac{x^2 y}{x^4+y^2}$.

So, $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$ doesn't exist.

Solution (iii):
Let $x = rcos\theta$ and $y = rsin\theta$.
$x^2 + y^2 = r^2$,
$xy(x^2 - y^2) = r^4 sin\theta cos\theta(cos^2\theta - sin^2\theta) = \frac{1}{2}r^4 \cdot 2sin\theta cos\theta \cdot cos2\theta$
$= \frac{1}{2}r^4 \cdot sin2\theta \cdot cos2\theta = \frac{1}{4}r^4 sin4\theta$
$\lim_{(x,y)\to(0,0)} \frac{xy(x^2 - y^2)}{x^2 + y^2} = \lim_{r\to 0^+} \frac{\frac{1}{4}r^4 sin4\theta}{r^2} = \frac{1}{4}\lim_{r\to 0^+} r^2 sin4\theta = 0$
Notes:
(i) $(x, y) \to (0,0) \Leftrightarrow \sqrt{x^2 + y^2} \to 0^+ \Leftrightarrow r \to 0^+$
(ii) As $|sin4\theta| \leq 1$, $|r^2 sin4\theta| \leq r^2$.
$\lim_{r\to 0^+} r^2 = 0 \Longrightarrow \lim_{r\to 0^+} r^2 sin4\theta = 0$

Exercise:
Show that $\lim_{(x,y)\to(0,0)} \frac{sin(x^2+y^2)}{x^2+y^2} = 1$.
[Hint: $\lim_{\theta\to 0} \frac{sin\theta}{\theta} = 1$]



Rules for Finding Limits:
Let $\phi \neq S \subset R^n$ and S is an open set. Let $f: S \to R$ and $g: S \to R$ be functions. Let $L, M, \lambda \in R$ and $X, P \in S$.
Suppose $\lim_{X\to P} f(X) = L$ and $\lim_{X\to P} g(X) = M$.
Then,
(i) $\lim_{X\to P}\big(f(X) + g(X)\big) = L + M$
(ii) $\lim_{X\to P}\big(f(X) - g(X)\big) = L - M$
(iii) $\lim_{X\to P}\big(f(X) \cdot g(X)\big) = LM$
(iv) $\lim_{X\to P} \frac{f(X)}{g(X)} = \frac{L}{M}$
(Assumed $M \neq 0$ and we can find $r \in R$ with $r > 0$ such that $B(P, r) \subset S$ and $g(X) \neq 0$ for any $X \in B(P, r)\backslash\{P\}$.)

(v) $\lim_{X\to P} \lambda f(X) = \lambda L$
Proof: Omitted (As Exercises)

Example 1 (re-visited)
Let $f: R^2 \to R$ be defined by $f(x, y) = xy$ and $(a, b) = (2,3)$. Show that $\lim_{(x,y)\to(a,b)} f(x, y) = 6$.
Proof:
Let $g: R^2 \to R$ be defined by $g(x, y) = x$ and $h: R^2 \to R$ be defined by $h(x, y) = y$.
$\lim_{(x,y)\to(2,3)} g(x, y) = \lim_{(x,y)\to(2,3)} x = \lim_{x\to 2} x = 2$ (Note: $(x, y) \to (2,3) \Longrightarrow x \to 2$)
Similarly, $\lim_{(x,y)\to(2,3)} h(x, y) = \lim_{(x,y)\to(2,3)} y = 3$.
$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(2,3)} g(x, y) \times \lim_{(x,y)\to(2,3)} h(x, y) = 2 \times 3 = 6.$

Example 2:
Suppose $f: R^2 \to R$ is a polynomial in x and y, say $f(x,y) = \sum_{(i,j)\in T} a_{(i,j)} x^i y^j$
where $a_{(i,j)} \in R$ for all $(i,j) \in T$, $(i,j) \in T \Longrightarrow i,j \in \{0,1,2,\cdots\}$ and T is a finite set.
We can show that $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 3:
Let $f: R^2 \to R$ be defined by $f(x,y) = 2x^4y^2 - 7xy + 4x^2y^3 - 5$. Find $\lim_{(x,y)\to(-1,2)} f(x,y)$.

Solution
$\lim_{(x,y)\to(-1,2)} f(x,y) = f(-1,2) = 8 + 14 + 32 - 5 = 49$

 # Continuity
Recall:
One Dimensional Case:
Let f be a function on $x \in R$ and let $a \in R$.
Suppose:
(i) $(a-\delta, a+\delta) \subset$ the domain of f for some $\delta > 0$
(that is, f is defined at all the points in a neighborhood of a.) **AND**
(ii) $\lim_{x\to a} f(x)$ exists as a real number **AND**
(iii) $\lim_{x\to a} f(x) = f(a)$.
Then, we say ***f* is continuous at *a***. Otherwise, we say f is NOT continuous at a or f is discontinuous at a.

Two Dimensional Case:
Let f be a function on $(x,y) \in R^2$ and let $(a,b) \in R^2$.
Suppose:
(i) $B\big((a,b),\delta\big) \subset$ the domain of f for some $\delta > 0$
(that is, f is defined at all the points in a neighborhood of (a,b).) **AND**
(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists as a real number **AND**
(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
Then, we say ***f* is continuous at (a,b)**. Otherwise, we say f is NOT continuous at (a,b) or f is discontinuous at (a,b).

$n-$ Dimensional Case:
Let f be a function on $X \in R^n$ and let $P \in R^n$.
Suppose:
(i) $B(P,\delta) \subset$ the domain of f for some $\delta > 0$
(that is, f is defined at all the points in a neighborhood of P.) **AND**
(ii) $\lim_{X\to P} f(X)$ exists as a real number **AND**
(iii) $\lim_{X\to P} f(X) = f(P)$.
Then, we say ***f* is continuous at *P***. Otherwise, we say f is NOT continuous at P or f is discontinuous at P.

One Dimensional Case:
Let $\phi \neq S \subset R$. Let f be a function on $x \in R$ and is defined on S.
We say f is continuous on S if f is continuous at x for any $x \in S$.

Two Dimensional Case:
Let $\phi \neq S \subset R^2$. Let f be a function on $(x,y) \in R^2$ and is defined on S.
We say f is continuous on S if f is continuous at (x,y) for any $(x,y) \in S$.

$n-$ Dimensional Case:
Let $\phi \neq S \subset R^n$. Let f be a function on $X \in R^n$ and is defined on S.
We say f is continuous on S if f is continuous at X for any $X \in S$.

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

<u>Example 1:</u>
Let $f: D \to R$ be defined by $f(x,y) = 1$ where $D = \{(x,y) \in R^2 : x^2 + y^2 \leq 1\}$. Show that f is continuous on D.

Proof:
For any $(a,b) \in D$, $\lim_{(x,y)\to(a,b)} f(x,y) = 1 = f(a,b)$.
So, f is continuous at (a,b).
Thus, f is continuous on D.

<u>Example 2:</u>
Let $g: R^2 \to R$ be defined by $g(x,y) = \begin{cases} 1 & if\ (x,y) \in D \\ 0 & otherwise \end{cases}$, where $D = \{(x,y) \in R^2 : x^2 + y^2 \leq 1\}$.
Show that g is NOT continuous on R^2.

Proof:
Suffices to show that g is NOT continuous at $(1,0)$.
$g(1,0) = 1$.
$\lim_{\substack{(x,y)\to(1,0) \\ x<1\ and\ y=0}} g(x,y) = \lim_{\substack{(x,y)\to(1,0) \\ x<1\ and\ y=0}} 1 = 1$
$\lim_{\substack{(x,y)\to(1,0) \\ x>1\ and\ y=0}} g(x,y) = \lim_{\substack{(x,y)\to(1,0) \\ x>1\ and\ y=0}} 0 = 0$
Thus, $\lim_{\substack{(x,y)\to(1,0) \\ y=0}} g(x,y)$ doesn't exist. Hence, $\lim_{(x,y)\to(1,0)} g(x,y)$ doesn't exist.

<u>Rules for Continuous Functions:</u>
Let $\phi \neq S \subset R^n$ and S is an open set. Let $f: S \to R$ and $g: S \to R$ be functions. Let $\lambda \in R$ and $P \in S$.
Suppose f and g are continuous at P.
Then,
(i) $f + g$ is continuous at P
(ii) $f - g$ is continuous at P
(iii) $f \cdot g$ is continuous at P
(iv) $\frac{f}{g}$ is continuous at P
(Assumed that we can find $r \in R$ with $r > 0$ such that $B(P,r) \subset S$ and $g(X) \neq 0$ for any $X \in B(P,r)$.)
(v) λf is continuous at P
Proof: Omitted (As Exercises)

<u>Theorem (Composition of Continuous Functions)</u>
Let $\phi \neq S \subset R^n$ and S is an open set. Let $\phi \neq I \subset R$ and I is an open interval.
Let $f: S \to R$ and $g: I \to R$ be functions. Let $P \in S$ and $f(P) \in I$.
Suppose f is continuous at P and g is continuous at $f(P)$.
Then, $g \circ f$ is continuous at P.
Proof: Omitted (As Exercise)

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

Example:

Show that $z = sin(x^2 + y^2)$ is continuous on R^2.

Proof:
Let $f: R^2 \to R$ be defined by $f(x, y) = x^2 + y^2$ and $g: R \to R$ be defined by $g(\theta) = sin\theta$.
As f is continuous on R^2 and g is continuous on R, $z = sin(x^2 + y^2) = g^\circ f(x, y)$ is continuous on R^2.

 # Partial Differentiation (Two Dimensional Case):
Let $\phi \neq S \subset R^2$ and S is an open set. Let $f: S \to R$ be a function on (x, y) and $(a, b) \in S$.
We define:

(i) $$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x + h, y) - f(x, y)}{h}$$

(ii) $$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)} = \lim_{h\to 0}\frac{f(a + h, b) - f(a, b)}{h}$$

(iii) $$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x, y + k) - f(x, y)}{k}$$

(iv) $$f_y(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)} = \lim_{k\to 0}\frac{f(a, b + k) - f(a, b)}{k}$$

Rules for finding partial derivative:

(i) To find $\frac{\partial f}{\partial x}$, regard y as a constant and differentiate with respect to x

(ii) To find $\frac{\partial f}{\partial y}$, regard x as a constant and differentiate with respect to y

Example 1:

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the function $f(x, y) = x^2 + 2xy^2 - y^3$.

Solutions

$\frac{\partial f}{\partial x} = 2x + 2y^2$ and $\frac{\partial f}{\partial y} = 4xy - 3y^2$.

Example 2:

Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = (x^2 + y^2)e^{-xy}$.

Solutions

$\frac{\partial z}{\partial x} = 2xe^{-xy} + (x^2 + y^2)e^{-xy} \cdot (-y) = (2x - x^2y - y^3)e^{-xy}$

$\frac{\partial z}{\partial y} = 2ye^{-xy} + (x^2 + y^2)e^{-xy} \cdot (-x) = (2y - xy^2 - x^3)e^{-xy}$.

Example 3:

The volume V (in cubic centimetres (or cm^3)) of 1 mole (or $mol.$) of an ideal gas is given by $V = \frac{82.06}{p}T$, where p is the pressure (in atmospheres (or atm)) and T is the absolute temperature (in Kelvins (or K)).

Find the rates of change of the volume of 1 $mol.$ of an ideal gas with respect to pressure (assuming temperature is kept constant) and with respect to temperature (assuming pressure is kept constant) when $T = 300K$ and $p = 5\ atm$.

Solutions

$V = \frac{82.06}{p} T$

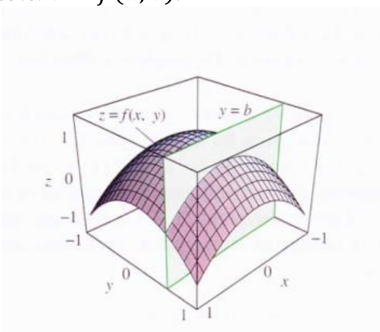
$\frac{\partial V}{\partial T} = \frac{82.06}{p}, \left.\frac{\partial V}{\partial T}\right|_{T=300, p=5} = \frac{82.06}{5} = 16.412$ (in cm^3/K)

$\frac{\partial V}{\partial p} = \frac{-82.06}{p^2} T, \left.\frac{\partial V}{\partial p}\right|_{T=300, p=5} = \frac{-82.06}{5^2} \times 300 = -984.72$ (in cm^3/atm)

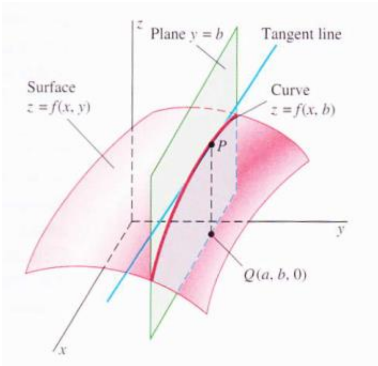
Negative sign means decreasing.
Positive sign means increasing.

Geometric Interpretation of Partial Derivatives

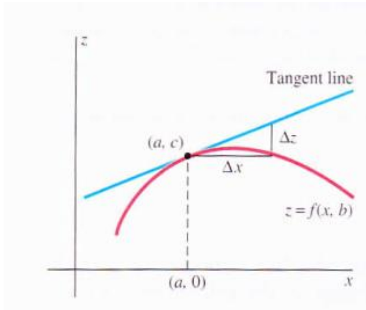
The value $f_x(a,b) = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$ is the slope of the tangent at the point $P(a,b,c)$ to the $x-$ curve through P on the surface $z = f(x,y)$.
Note: $c = f(a,b)$.



A vertical plane parallel to the $xz-$ plane intersects the surface $z = f(x,y)$ in an $x-$ curve.



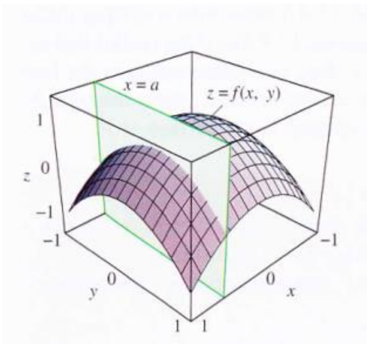
An $x-$ curve and its tangent at P



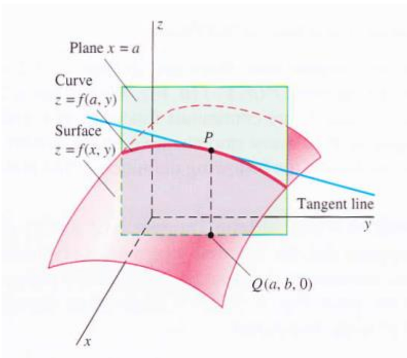
Projection into the $xz-$ plane of the $x-$ curve through $P(a,b,c)$ and its tangent line

The value $f_y(a,b) = \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)} = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$ is the slope of the tangent at the point $P(a,b,c)$ to the $y-$ curve through P on the surface $z = f(x,y)$.

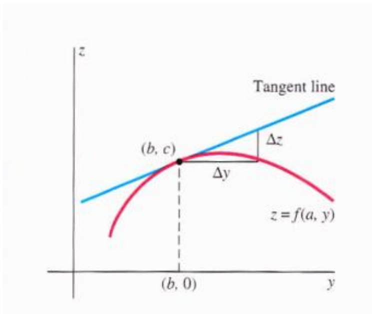
Note: $c = f(a,b)$.



A vertical plane parallel to the $yz-$ plane intersects the surface $z = f(x,y)$ in a $y-$ curve.



A $y-$ curve and its tangent at P



Projection into the $yz-$ plane of the $y-$ curve through $P(a,b,c)$ and its tangent line

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

One Dimensional Case (The Line Tangent to a Curve)

Let $\phi \neq S \subset R$ and S is an open interval. Let $f: S \rightarrow R$ be a function on x and $a \in S$. Let $y = f(x)$.
Suppose f is differentiable on S.
An equation of the tangent to the curve $y = f(x)$ is
$\frac{y-f(a)}{x-a} = f'(a)$.
That is, $y = f(a) + f'(a)(x-a)$.

Two Dimensional Case (The Plane Tangent to a Surface)

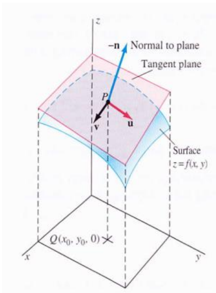
Let $\phi \neq S \subset R^2$ and S is an open set. Let $f: S \rightarrow R$ be a function on (x, y) and $(a, b) \in S$.
Let $z = f(x, y)$. Suppose we can find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ on S.
Let us define $C_1: (a-\delta, a+\delta) \rightarrow R^3$ by $C_1(t) = \big(t, b, f(t, b)\big)$.
C_1 is a curve passing through $\big(a, b, f(a, b)\big)$ on the surface $z = f(x, y)$.
$\vec{u} = C_1{}'(a)$ is a vector on the tangent plane to the surface $z = f(x, y)$ at $\big(a, b, f(a, b)\big)$.
$\vec{u} = C_1{}'(a) = \big(1, 0, f_x(a, b)\big)$.
Let us define $C_2: (b-\delta, b+\delta) \rightarrow R^3$ by $C_2(t) = \big(a, t, f(a, t)\big)$.
C_2 is a curve passing through $\big(a, b, f(a, b)\big)$ on the surface $z = f(x, y)$.
$\vec{v} = C_2{}'(b)$ is a vector on the tangent plane to the surface $z = f(x, y)$ at $\big(a, b, f(a, b)\big)$.
$\vec{v} = C_2{}'(b) = \Big(0, 1, f_y(a, b)\Big)$.
Let $\vec{n} = \vec{u} \times \vec{v}$. Then, $\vec{n}$ will be a normal vector of required tangent plane.

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(a, b) \\ 0 & 1 & f_y(a, b) \end{vmatrix} = -f_x(a, b)\vec{i} - f_y(a, b)\vec{j} + \vec{k} = \big(-f_x(a, b), -f_y(a, b), 1\big)$$

An equation of the plane tangent to the surface $z = f(x, y)$ at (a, b, c) is
$\big((x, y, z) - (a, b, c)\big) \cdot \big(-f_x(a, b), -f_y(a, b), 1\big) = 0$
$-f_x(a, b)(x-a) - f_y(a, b)(y-b) + z - c = 0$
$z = c + f_x(a, b)(x-a) + f_y(a, b)(y-b)$

Remark:

$z = f(x, y)$
$\frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} = f_x(a, b)$ and $\frac{\partial z}{\partial y}\Big|_{(x,y)=(a,b)} = f_y(a, b)$



Summary:

An equation of the plane tangent to the surface $z = f(x, y)$ at $\big(a, b, f(a, b)\big)$ is
$\boldsymbol{z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)}$

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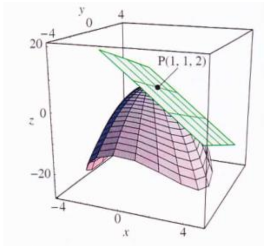
Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

Example 1:

Write an equation of the plane tangent to the paraboloid $z = 5 - 2x^2 - y^2$ at the point $P(1,1,2)$.



Solutions

$z = f(x,y) = 5 - 2x^2 - y^2$

$\frac{\partial z}{\partial x} = -4x, \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(1,1)} = -4$

$\frac{\partial z}{\partial y} = -2y, \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(1,1)} = -2$

A normal vector of required tangent plane is $\left(-f_x(1,1), -f_y(1,1), 1\right) = (4,2,1)$.

An equation of required tangent is

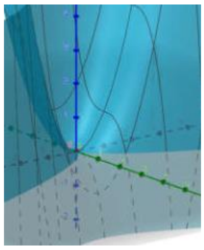
$\left((x,y,z) - (1,1,2)\right) \cdot (4,2,1) = 0$

$4(x-1) + 2(y-1) + z - 2 = 0$

$z = -4x - 2y + 8$

Example 2:

Write an equation of the plane tangent to the paraboloid $z = x^2 - y^3$ at the point $P(2,1,3)$.



Solutions

$z = f(x,y) = x^2 - y^3$

$\frac{\partial z}{\partial x} = 2x, \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(2,1)} = 4$

$\frac{\partial z}{\partial y} = -3y^2, \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(2,1)} = -3$

A normal vector of required tangent plane is $\left(-f_x(2,1), -f_y(2,1), 1\right) = (-4,3,1)$.

An equation of required tangent is

$\left((x,y,z) - (2,1,3)\right) \cdot (-4,3,1) = 0$

$-4(x-2) + 3(y-1) + z - 3 = 0$

$z = 4x - 3y - 2$

Course Code: **MATH 2000**
Course Name: **Engineering Mathematics I**

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

 # Partial Differentiation (Three Dimensional Case):
Let $\phi \neq S \subset R^3$ and S is an open set.
Let $f: S \to R$ be a function on (x, y, z) and $(a, b, c) \in S$.
We define:

(i) $$f_x(x,y,z) = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y,z) - f(x,y,z)}{h}$$

(ii) $$f_x(a,b,c) = \left.\frac{\partial f}{\partial x}\right|_{(x,y,z)=(a,b,c)} = \lim_{h\to 0}\frac{f(a+h,b,c) - f(a,b,c)}{h}$$

(iii) $$f_y(x,y,z) = \frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x,y+k,z) - f(x,y,z)}{k}$$

(iv) $$f_y(a,b,c) = \left.\frac{\partial f}{\partial y}\right|_{(x,y,z)=(a,b,c)} = \lim_{k\to 0}\frac{f(a,b+k,c) - f(a,b,c)}{k}$$

(v) $$f_z(x,y,z) = \frac{\partial f}{\partial z} = \lim_{l\to 0}\frac{f(x,y,z+l) - f(x,y,z)}{l}$$

(vi) $$f_z(a,b,c) = \left.\frac{\partial f}{\partial z}\right|_{(x,y,z)=(a,b,c)} = \lim_{l\to 0}\frac{f(a,b,c+l) - f(a,b,c)}{l}$$

Rules for finding partial derivative:

(i) To find $\frac{\partial f}{\partial x}$, regard y and z as constants and differentiate with respect to x

(ii) To find $\frac{\partial f}{\partial y}$, regard x and z as constants and differentiate with respect to y

(iii) To find $\frac{\partial f}{\partial z}$, regard x and y as constants and differentiate with respect to z

Example:

Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ of the function $f(x,y,z) = x^2y^3z^4$.

Solutions

$\frac{\partial f}{\partial x} = 2xy^3z^4$, $\frac{\partial f}{\partial y} = 3x^2y^2z^4$ and $\frac{\partial f}{\partial z} = 4x^2y^3z^3$

 # Partial Differentiation (n −Dimensional Case):
Let $\phi \neq S \subset R^n$ and S is an open set.
Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$ and $A = (a_1, a_2, \cdots, a_n) \in S$.

We define $e_i = \begin{cases} x_i & if \ \ i \neq j \\ x_i + h & if \ \ i = j \end{cases}$ and

$f_{x_j}(X) = f_{x_j}(x_1, x_2, \cdots, x_n) = \frac{\partial f}{\partial x_j} = \lim_{h\to 0}\frac{f(e_1,e_2,\cdots,e_n) - f(x_1,x_2,\cdots,x_n)}{h}$.

We define $\theta_i = \begin{cases} a_i & if \ \ i \neq j \\ a_i + h & if \ \ i = j \end{cases}$ and

$f_{x_j}(A) = f_{x_j}(a_1, a_2, \cdots, a_n) = \left.\frac{\partial f}{\partial x_j}\right|_{X=A} = \lim_{h\to 0}\frac{f(\theta_1,\theta_2,\cdots,\theta_n) - f(a_1,a_2,\cdots,a_n)}{h}$.

Rule for finding partial derivative:
To find $\frac{\partial f}{\partial x_j}$, regard x_i $(i \neq j)$ as constants and differentiate with respect to x_j

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Example 1:

Find the four partial derivatives of the function $g(x, y, u, v) = e^{ux} sinvy$.

Solutions

$g_x = ue^{ux} sinvy$, $g_u = xe^{ux} sinvy$, $g_y = ve^{ux} cosvy$, $g_v = ye^{ux} cosvy$.

Example 2:

Find the four partial derivatives of the function $g(x, y, u, v) = x^2y^3 - u^4v^5$.

Solutions

$g_x = 2xy^3$, $g_y = 3x^2y^2$, $g_u = -4u^3v^5$, $g_v = -5u^4v^4$.

 # Higher Order Partial Derivatives

We define:

(i) $$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

(ii) $$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

(iii) $$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

(iv) $$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

(v) $$f_{xxy} = (f_{xx})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y \partial x^2}$$

(vi) $$f_{xyx} = (f_{xy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$$

(vii) $$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and others.

Example:

Show that the partial derivatives of third and fourth orders of the function $z = f(x, y) = x^2 + 2xy^2 - y^3$ are constants.

Solutions

$f_x = 2x + 2y^2$; $f_y = 4xy - 3y^2$;
$f_{xx} = 2$; $f_{xy} = 4y$; $f_{yx} = 4y$; $f_{yy} = 4x - 6y$;
$f_{xxx} = 0$; $f_{xxy} = 0$; $f_{xyx} = 0$; $f_{xyy} = 4$; $f_{yxx} = 0$; $f_{yxy} = 4$; $f_{yyx} = 4$; $f_{yyy} = -6$
Partial derivatives of fourth orders are all zeros.
So, partial derivatives of third and fourth orders are constants.

Remark

In general, f_{xy} and f_{yx} may not be the same.
We can show that if f_{xy} and f_{yx} are continuous on an open set, then $f_{xy} = f_{yx}$.

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

<u># Multivariable Optimization Problem</u>
<u># Global Minima and Global Maxima</u>

Let $\phi \neq S \subset R^n$. Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$ and is defined on S. Let $m, M \in R$.
We say f attains **<u>the</u> global minimum value** (or **<u>the</u> absolute minimum value**) m on S if:
(i) $f(X) \geq m$ for any $X \in S$ **AND**
(ii) we can find $U \in S$ such that $f(U) = m$

We say f attains **<u>the</u> global maximum value** (or **<u>the</u> absolute maximum value**) M on S if:
(i) $f(X) \leq M$ for any $X \in S$ **AND**
(ii) we can find $V \in S$ such that $f(V) = M$

Remark: We say $(U, f(U))$ **<u>a</u> global minima** and $(V, f(V))$ **<u>a</u> global maxima**.

<u>Theorem 1:</u>
Let $m_1, m_2 \in R$. Suppose $f(X) \geq m_1$ for any $X \in S$ **AND** $f(X) \geq m_2$ for any $X \in S$.
Suppose we can find $U_1, U_2 \in S$ such that $f(U_1) = m_1$ and $f(U_2) = m_2$.
Then, $m_1 = m_2$.
Proof:
$m_2 = f(U_2) \geq m_1$. Also, $m_1 = f(U_1) \geq m_2$. So, $m_1 = m_2$.

<u>Theorem 2:</u>
Let $M_1, M_2 \in R$. Suppose $f(X) \leq M_1$ for any $X \in S$ **AND** $f(X) \leq M_2$ for any $X \in S$.
Suppose we can find $V_1, V_2 \in S$ such that $f(V_1) = M_1$ and $f(V_2) = M_2$.
Then, $M_1 = M_2$.
Proof: Omitted (As Exercise)

<u>Theorem:</u>
Let $\phi \neq S \subset R^n$. Suppose S is **<u>closed and bounded</u>**.
Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$ and is defined on S.
Suppose f is **<u>continuous on S</u>**.
Then, f must attain the global minimum value and the global maximum value on S.
Proof: Will be discussed on course "Real Analysis"

<u>Definition</u>
$(W, f(W))$ is called a global extrema if it is a global maxima or it is a global minima

<u># Local Minima and Local Maxima</u>
Let $\phi \neq S \subset R^n$. Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$ and is defined on S. Let $U, V \in S$.

We say $(U, f(U))$ **<u>a</u> local minima** (or **<u>a</u> relative minima**) if we can find $r \in R$ with $r > 0$ such that $B(U, r) \subset S$ and $f(X) \geq f(U)$ for any $X \in B(U, r)$. In this case, $f(U)$ is called **<u>a</u> local minimum value** (or **<u>a</u> relative minimum value**).

We say $(V, f(V))$ **<u>a</u> local maxima** (or **<u>a</u> relative maxima**) if we can find $r \in R$ with $r > 0$ such that $B(V, r) \subset S$ and $f(X) \leq f(V)$ for any $X \in B(V, r)$. In this case, $f(V)$ is called **<u>a</u> local maximum value** (or **<u>a</u> relative maximum value**).

<u>Theorem 1:</u>
Let $U \in S$ and U is an interior point of S.
$(U, f(U))$ is **<u>a global mínima</u>** $\Rightarrow$ $(U, f(U))$ is **<u>a local minima</u>**

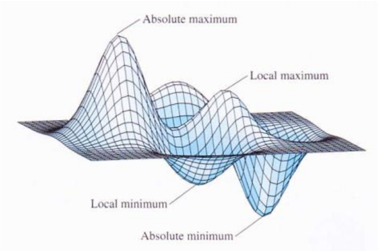
Theorem 2:
Let $V \in S$ and V is an interior point of S.
$(V, f(V))$ is **a global maxima** $\Longrightarrow (V, f(V))$ is **a local maxima**

Remark: The converse of above theorems may not be true.

Diagram showing the relationship between global maxima and local maxima / between global minima and local minima (Note: the choice of the region / boundary is important.)
$f(x,y) = 3(x-1)^2 e^{-x^2-(y+1)^2} + (-2x + 10x^3 + 10y^5)e^{-x^2-y^2} - \frac{1}{3}e^{-(x+1)^2-y^2}$ for
$(x,y) \in \{(a,b) \in R^2: -3 \le a \le 3, -3 \le b \le 3\}$.

A local maximum value MAY not be the global maximum value.

A local minimum value MAY not be the global minimum value.

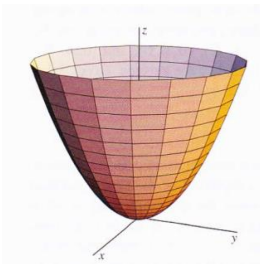


Definition
$(W, f(W))$ is called a local extrema if it is a local maxima or it is a local minima

Theorem 1 (Necessary Conditions for Local Minima)
Let $\phi \neq S \subset R^n$. Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$ and is defined on S. Let $U \in S$ and $r \in R$ with $r > 0$.
Suppose $B(U,r) \subset S$ and $f(X) \ge f(U)$ for any $X \in B(U,r)$.
That is, $(U, f(U))$ is **a local minima**.
Suppose we can find $f_{x_j}(X)$ for any $X \in B(U,r)$ and $j = 1,2,\cdots,n$.
Then, $f_{x_j}(U) = 0$ for $j = 1,2,\cdots,n$

Theorem 2 (Necessary Conditions for Local Maxima)
Let $\phi \neq S \subset R^n$. Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$ and is defined on S. Let $V \in S$ and $r \in R$ with $r > 0$.
Suppose $B(V,r) \subset S$ and $f(X) \le f(V)$ for any $X \in B(V,r)$.
That is, $(V, f(V))$ is **a local minima**.
Suppose we can find $f_{x_j}(X)$ for any $X \in B(V,r)$ and $j = 1,2,\cdots,n$.
Then, $f_{x_j}(V) = 0$ for $j = 1,2,\cdots,n$

Example 1:
Let $f: R^2 \to R$ be defined by $f(x,y) = x^2 + y^2$.
Show that $(0,0)$ is a local minima and is the global minima on
$D = \{(x,y) \in R^2: x^2 + y^2 \le 1\}$.

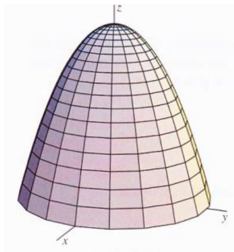


Proof:

$f(0,0) = 0 \le x^2 + y^2 = f(x, y)$ for any $(x, y) \in B(O, 1)$, so it is a local minima.
$f(0,0) = 0 \le x^2 + y^2 = f(x, y)$ for any $(x, y) \in D$, so it is a global minima on D.
As $x^2 + y^2 = 0 \Leftrightarrow (x, y) = (0,0)$, it is the global minima on D.

Exercise 1:

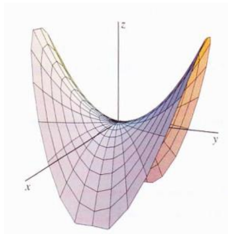
Let $f: R^2 \to R$ be defined by $f(x, y) = 1 - x^2 - y^2$.
Show that $(0,0)$ is a local maxima and is the global maxima on
$D = \{(x, y) \in R^2 : x^2 + y^2 \le 1\}$.



Proof: Omitted (As Exercise)

Exercise 2:

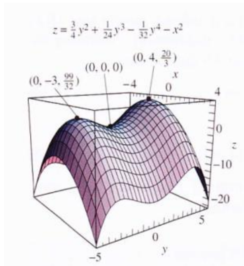
Let $f: R^2 \to R$ be defined by $f(x, y) = y^2 - x^2$.
Show that $(0,0)$ is **neither a local maxima nor a local minima**.
This point is called a **saddle point**.



Proof: Omitted (As Exercise)

Example 2:

Find all points on the surface $z = \frac{3}{4}y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 - x^2$ at which the tangent plane is horizontal.



Solutions

$z_x = -2x$
Put $z_x = 0$, we have $x = 0$.
$z_y = \frac{3}{2}y + \frac{1}{8}y^2 - \frac{1}{8}y^3 = \frac{-1}{8}y(y^2 - y - 12) = \frac{-1}{8}y(y-4)(y+3)$
Put $z_y = 0$, we have $y = 0$ or 4 or -3.
Required points are $(0,0,0)$, $\left(0,4,\frac{20}{3}\right)$ and $\left(0,-3,\frac{99}{32}\right)$

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

Strategy for finding global extrema:

Let $\phi \neq S \subset R^n$

Usual Case: A continuous function f on closed and bounded region S in R^n **AND** $f_{x_j}(X)$ exists for all $X \in S\backslash\partial S$

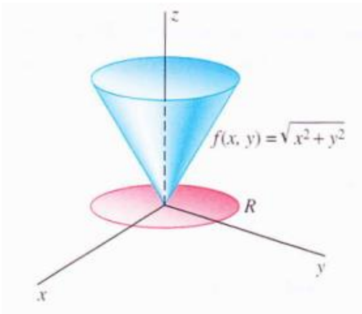
(i) Find $M_{\partial S} = max\{f(X): X \in \partial S\}$ and $m_{\partial S} = min\{f(X): X \in \partial S\}$

(ii) Consider $T = \{ \quad X \in S\backslash\partial S: f_{x_j}(X) = 0$ for $j = 1,2,\cdots,n \quad \}$
and find $M_{S\backslash\partial S} = max\{f(X): X \in T\}$ and $m_{S\backslash\partial S} = min\{f(X): X \in T\}$

(iii) The global maximum value is $max\{M_{\partial S}, M_{S\backslash\partial S}\}$
The global minimum value is $min\{m_{\partial S}, m_{S\backslash\partial S}\}$

Example 1:

Let $f(x,y) = \sqrt{x^2+y^2}$ on the region R consisting of the points on and within the circle $x^2 + y^2 = 1$ in the $xy -$ plane. Find the global maximum value and the global minimum value of f on R.



Solutions

(i) When $x^2 + y^2 = 1$, $f(x,y) = \sqrt{x^2+y^2} = 1$.
So, $M_{\partial R} = max\{f(X): X \in \partial R\} = 1$ and $m_{\partial R} = min\{f(X): X \in \partial R\} = 1$

(ii) $f(x,y) = \sqrt{x^2+y^2}$
For $x^2 + y^2 > 0$,
$$f_x(x,y) = \frac{1}{2\sqrt{x^2+y^2}} \cdot (2x) = \frac{x}{\sqrt{x^2+y^2}}$$
$$f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$
$f_x(x,y) = 0 \Leftrightarrow x = 0$
$f_y(x,y) = 0 \Leftrightarrow y = 0$
But $(0,0)$ doesn't satisfy $x^2 + y^2 > 0$.
Thus, $\{ \quad X \in \{(x,y) \in R^2: 0 < x^2 + y^2 < 1\}: f_{x_j}(X) = 0$ for $j = 1,2,\cdots,n \quad \} = \phi$
$f(0,0) = \sqrt{0^2+0^2} = 0$

(iii) The global maximum value is $max\{1,0\} = 1$
The global minimum value is $min\{1,0\} = 0$

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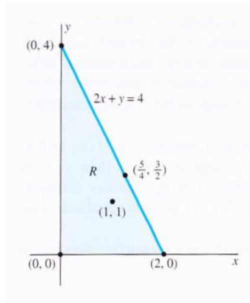
Example 2:

Find the maximum and minimum values attained by the function $f(x,y) = xy - x - y + 3$ at points of the triangular region R in the $xy-$ plane with vertices at $(0,0)$, $(2,0)$ and $(0,4)$.



Solutions

(i) When $x = 0, f(0,y) = -y + 3$

$max\{f(X): X \in \partial R \; and \; x = 0\} = 0 + 3 = 3$

$min\{f(X): X \in \partial R \; and \; x = 0\} = -4 + 3 = -1$

When $y = 0, f(x,0) = -x + 3$

$max\{f(X): X \in \partial R \; and \; y = 0\} = 0 + 3 = 3$

$min\{f(X): X \in \partial R \; and \; y = 0\} = -2 + 3 = 1$

When $2x + y = 4$,

$f(x,y) = x(4 - 2x) - x - (4 - 2x) + 3$

$= -2x^2 + 5x - 1 = -2\left(x^2 - \frac{5}{2}x\right) - 1$

$= -2\left(x - \frac{5}{4}\right)^2 + \frac{17}{8}$

When $x = 0, y = 4, \; f(0,4) = -1$.

When $x = 2, y = 0, \; f(2,0) = 1$.

$max\{f(X): X \in \partial R \; and \; 2x + y = 4\} = max\left\{-1,1,\frac{17}{8}\right\} = \frac{17}{8}$

$min\{f(X): X \in \partial R \; and \; 2x + y = 4\} = min\left\{-1,1,\frac{17}{8}\right\} = -1$

So, $M_{\partial R} = max\{f(X): X \in \partial R\} = max\left\{3,3,\frac{17}{8}\right\} = 3$

and $m_{\partial R} = min\{f(X): X \in \partial R\} = min\{-1,1,-1\} = -1$

(ii) $f(x,y) = xy - x - y + 3$

$f_x(x,y) = y - 1$

$f_y(x,y) = x - 1$

$f_x(x,y) = 0 \Leftrightarrow x = 1$

$f_y(x,y) = 0 \Leftrightarrow y = 1$

$(1,1) \in R \backslash \partial R$

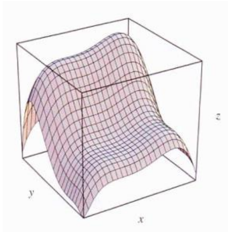
$f(1,1) = 1 - 1 - 1 + 3 = 2$

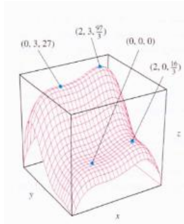
(iii) The global maximum value is $max\{3,2\} = 3$

The global minimum value is $min\{-1,2\} = -1$

<u>Example 3:</u>

Find the highest point on the surface
$z = f(x, y) = \frac{8}{3}x^3 + 4y^3 - x^4 - y^4$.



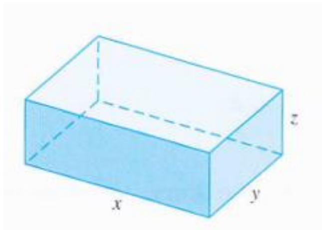


<u>Solutions</u>

$f_x(x, y) = 8x^2 - 4x^3 = 4x^2(2 - x)$
$f_x(x, y) = 0 \Leftrightarrow 4x^2(2 - x) = 0 \Leftrightarrow x = 0$ or 2
$f_y(x, y) = 12y^2 - 4y^3 = 4y^2(3 - y)$
$f_y(x, y) = 0 \Leftrightarrow 4y^2(3 - y) = 0 \Leftrightarrow y = 0$ or 3
For $f_x(x, y) = 0$ and $f_y(x, y) = 0$, we have only 4 points $(0,0)$, $(0,3)$, $(2,0)$ and $(2,3)$ for consideration.
$f(0,0) = 0$, $f(0,3) = 27$, $f(2,0) = \frac{16}{3}$, $f(2,3) = \frac{97}{3}$
When $x \to +\infty$, $f(x, y) \to -\infty$ as it is dominated by $-x^4$.
When $x \to -\infty$, $f(x, y) \to -\infty$ as it is dominated by $-x^4$.
When $y \to +\infty$, $f(x, y) \to -\infty$ as it is dominated by $-y^4$.
When $y \to -\infty$, $f(x, y) \to -\infty$ as it is dominated by $-y^4$.
Thus, the highest point is $\left(2,3,\frac{97}{3}\right)$.

<u>Example 4:</u>

Find the minimum cost of a rectangular box with volume $48\ ft^3$ if the front and back cost $\$1/ft^2$, the top and bottom cost $\$2/ft^2$, and the two ends cost $\$3/ft^2$. This box is shown in the figure.



<u>Solutions</u>

Let the length be x ft., the width be y ft., the height be z ft. and the cost be $\$C(x,y)$.

Then, $C(x,y) = 4xy + \frac{96}{y} + \frac{288}{x}$ (Assumed both $x > 0$ and $y > 0$).

Note:

The cost is $\$2 \cdot (2xy + xz + 3yz) = \$(4xy + 2xz + 6yz)$ and $xyz = 48$.

$C_x(x,y) = 4y - 288x^{-2}$

$C_x(x,y) = 0 \Leftrightarrow 4y - 288x^{-2} = 0 \Leftrightarrow \dfrac{288}{x} = 4xy$

$C_y(x,y) = 4x - 96y^{-2}$

$C_y(x,y) = 0 \Leftrightarrow 4x - 96y^{-2} = 0 \Leftrightarrow \dfrac{96}{y} = 4xy$

For both $C_x(x,y) = 0$ and $C_x(x,y) = 0$, $\frac{288}{x} = 4xy = \frac{96}{y}$.

So, $y = \frac{1}{3}x$ and $x^3 = 216$. Thus $x = 6$ and $y = 2$ (so, $z = 4$)

$C(x,y) = 4xy + \dfrac{96}{y} + \dfrac{288}{x} = 12xy = 144$

The minimum cost is $\$144$ when the dimensions are $6\ ft. \times 2\ ft. \times 4\ ft.$

Note: We don't need to consider the boundary.

Choose $\delta, M \in R$ with $\delta > 0$ and $M > 0$.

Let $T = \{(x,y) \in R^2 : \delta < x < M\ and\ \delta < y < M\}$.

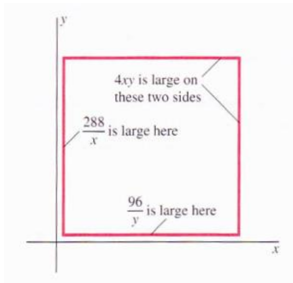
We can choose δ and M so that on the boundaries,

$\frac{96}{y} > 1000$ on the side nearest to $x-$ axis

$\frac{288}{x} > 1000$ on the side nearest to $y-$ axis

$4xy > 1000$ on the remaining two sides

So $C(x,y) > 1000$ on T



Example 5:

Determine whether the function $f(x,y,z) = xy + yz - xz$ has any local extrema.

Solutions

$f_x(x,y,z) = y - z$
$f_x(x,y,z) = 0 \Leftrightarrow y - z = 0 \Leftrightarrow y = z$
$f_y(x,y,z) = x + z$
$f_y(x,y,z) = 0 \Leftrightarrow x + z = 0 \Leftrightarrow x = -z$
$f_z(x,y,z) = y - x$
$f_z(x,y,z) = 0 \Leftrightarrow y - x = 0 \Leftrightarrow x = y$
Put $f_x(x,y,z) = 0$ and $f_y(x,y,z) = 0$ and $f_z(x,y,z) = 0$, we have
$x = y$ and $x = -z$ and $y = z$.
Thus, $x = y = z = 0$.
$f(0,0,0) = 0$
$f(t,t,t) = t^2 \geq 0 = f(0,0,0)$ for any $t \in R$.
$f(-t,t,-t) = -3t^2 \leq 0 = f(0,0,0)$ for any $t \in R$.
So, $(0,0,0)$ is neither a local maxima nor a local minima.
So, f has no local extrema on R^3.

Note:
$f(t,t,t) = t^2 \to +\infty$ as $t \to +\infty$
$f(t,t,t) = t^2 \to +\infty$ as $t \to -\infty$
$f(-t,t,-t) = -3t^2 \to -\infty$ as $t \to +\infty$
$f(-t,t,-t) = -3t^2 \to -\infty$ as $t \to -\infty$
So, f has no global extrema on R^3.

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

 # Increments and Linear Approximation
Recall:
One Dimensional Case:
Let $f: R \to R$ be a function on x.
Suppose f is differentiable at a.
So, $f(a+h) - f(a) \approx f'(a) \cdot h$ when $h \approx 0$.

Two Dimensional Case:
Let $f: R^2 \to R$ be a function on (x, y).
Suppose f_x and f_y are continuous at points near to (a, b). Let $z = f(x, y)$.

$f(a+h, b+k) - f(a, b+k) \approx f_x(a, b+k) \cdot h$ when $h \approx 0$
$f(a, b+k) - f(a, b) \approx f_y(a, b) \cdot k$ when $k \approx 0$

So, $f(a+h, b+k) - f(a, b) \approx f_x(a, b+k) \cdot h + f_y(a, b) \cdot k$ when $h \approx 0$ and $k \approx 0$

Assume f_x is continuous near to (a, b). Then, $f_x(a, b+k) \approx f_x(a, b)$ when $k \approx 0$.

Thus, when $h \approx 0$ and $k \approx 0$, we have $f(a+h, b+k)$
$\approx f(a, b) + f_x(a, b) \cdot h + f_y(a, b) \cdot k$
$= f(a, b) + \left(f_x(a, b), f_y(a, b)\right) \cdot (h, k)$

Note: $\Delta z = f(a+h, b+k) - f(a, b)$; $dx = \Delta x = h$; $dy = \Delta y = k$

We define $dz = f_x(a, b) \cdot dx + f_y(a, b) \cdot dy$.

Then, $f(a+h, b+k) - f(a, b) = \Delta z \approx dz = f_x(a, b) \cdot h + f_y(a, b) \cdot k$

For $z = f(x, y)$, at general point (x, y), $dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy$

Example 1:
Find the differential df of the function $f(x, y) = x^2 + 3xy - 2y^2$. Then, compare df and the actual increment Δf when (x, y) changes from $P(3,5)$ to $Q(3.2,4.9)$.

Solutions
$f_x(x, y) = 2x + 3y$; $f_y(x, y) = 3x - 4y$
$df = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy = (2x + 3y)dx + (3x - 4y)dy$

$f(3,5) = 4$; $f(3.2,4.9) = 9.26$;
$f_x(3,5) = 21$; $f_y(3,5) = -11$;
$dx = \Delta x = 3.2 - 3 = 0.2$; $dy = \Delta y = 4.9 - 5 = -0.1$.
For (x, y) changes from $P(3,5)$ to $Q(3.2,4.9)$,
$\Delta f = 9.26 - 4 = 5.26$
$df = 21 \times 0.2 + (-11) \times (-0.1) = 5.3$
$df \approx \Delta f$

Example 2:

Use linear approximation to estimate $\sqrt{2 \cdot (2.02)^3 + (2.97)^2}$.

Solutions

Let f be a real valued function on (x, y) and is defined by $f(x, y) = \sqrt{2x^3 + y^2}$.
(Note: We may assume $x \geq 0$ so that it is well defined.)
Let $z = f(x, y)$.
$f(2,3) = \sqrt{2 \times 8 + 9} = 5$;
$f_x(x, y) = \frac{1}{2\sqrt{2x^3+y^2}} \cdot 6x^2 = \frac{3x^2}{\sqrt{2x^3+y^2}}$; $f_x(2,3) = \frac{12}{5}$
$f_y(x, y) = \frac{1}{2\sqrt{2x^3+y^2}} \cdot 2y = \frac{y}{\sqrt{2x^3+y^2}}$; $f_y(2,3) = \frac{3}{5}$
$dx = \Delta x = 2.02 - 2 = 0.02$; $dy = \Delta y = 2.97 - 3 = -0.03$
$$dz = \frac{12}{5} \times 0.02 + \frac{3}{5} \times (-0.03) = 0.03$$
$$\sqrt{2 \cdot (2.02)^3 + (2.97)^2} \approx 5 + 0.03 = 5.03$$

Note: $\sqrt{2 \cdot (2.02)^3 + (2.97)^2} \approx 5.0305$ (by calculator)

Example 3:

The volume V (in cubic centimetres (or cm^3)) of 1 mole (or $mol.$) of an ideal gas is given by $V = \frac{82.06}{p} T$, where p is the pressure (in atmospheres (or atm)) and T is the absolute temperature (in Kelvins (or K)).

Approximate the change in V when p is increased from $5\ atm$ to $5.2\ atm$ and T is increased from $300K$ to $310K$.

Solutions

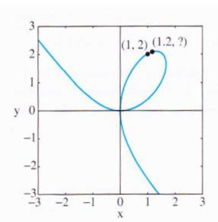
$V = \frac{82.06}{p} T$; When $T = 300$ and $p = 5$, $V = \frac{82.06}{5} \times 300 = 4923.6$ (in cm^3)
$\frac{\partial V}{\partial T} = \frac{82.06}{p}$, $\left.\frac{\partial V}{\partial T}\right|_{T=300,p=5} = \frac{82.06}{5} = 16.412$ (in cm^3/K)
$\frac{\partial V}{\partial p} = \frac{-82.06}{p^2} T$, $\left.\frac{\partial V}{\partial p}\right|_{T=300,p=5} = \frac{-82.06}{5^2} \times 300 = -984.72$ (in cm^3/atm)
$dp = \Delta p = 5.2 - 5 = 0.2$; $dT = \Delta T = 310 - 300 = 10$
$\Delta V \approx dV = 16.412 \times 10 + (-984.72) \times 0.2 = -32.824$ (in cm^3)

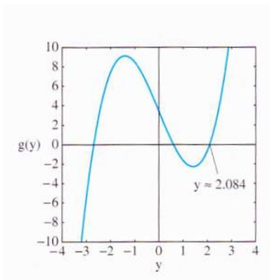
Note: $\Delta V = \frac{82.06}{5.2} \times 310 - \frac{82.06}{5} \times 300 = -31.5615$ (in cm^3)

Example 4:

The point $(1,2)$ lies on the curve with equation
$f(x, y) = 2x^3 + y^3 - 5xy = 0$.

Approximate the $y-$ coordinate of the nearby point (x, y) on this curve for which $x = 1.2$.





The graph of
$g(y) = y^3 - 6y + 3.456$
(Put $x = 1.2$ into
$2x^3 + y^3 - 5xy$)

Solutions
Let $z = f(x, y) = 2x^3 + y^3 - 5xy$, $f(1,2) = 0$ and $f(1.2, 2 + \Delta y) = 0$.
$\Delta z = 0$.
$f_x(x, y) = 6x^2 - 5y$; $f_x(1,2) = -4$; $dx = \Delta x = 1.2 - 1 = 0.2$
$f_y(x, y) = 3y^2 - 5x$; $f_y(1,2) = 7$; $dy = \Delta y$
$0 = \Delta z \approx dz = (-4) \times 0.2 + 7\Delta y$
So, $\Delta y \approx \frac{4\times 0.2}{7} \approx 0.114$
Required $y -$ coordinate $\approx 2 + 0.114 = 2.114$ (may take the approximate value 2.1)

Note: Required $y -$ coordinate ≈ 2.084 (by Newton's Method)

Three Dimensional Case:
Let $f: R^3 \to R$ be a function on (x, y, z).
Suppose f_x, f_y and f_z are continuous at points near to (a, b, c).

$f(a + h, b + k, c + l) - f(a, b + k, c + l) \approx f_x(a, b + k, c + l) \cdot h$ when $h \approx 0$
$f(a, b + k, c + l) - f(a, b, c + l) \approx f_y(a, b, c + l) \cdot k$ when $k \approx 0$
$f(a, b, c + l) - f(a, b, c) \approx f_z(a, b, c) \cdot l$ when $l \approx 0$

So, $f(a + h, b + k, c + l) - f(a, b, c)$
$\approx f_x(a, b + k, c + l) \cdot h + f_y(a, b, c + l) \cdot k + f_z(a, b, c) \cdot l$ when $h \approx 0$ and $k \approx 0$ and $l \approx 0$

Assume f_x and f_y are continuous near to (a, b, c).
Then, $f_x(a, b + k, c + l) \approx f_x(a, b, c)$ and $f_y(a, b, c + l) \approx f_y(a, b, c)$ when $k \approx 0$ and $l \approx 0$

Thus, when $h \approx 0$ and $k \approx 0$ and $l \approx 0$, we have $f(a + h, b + k, c + l)$
$\approx f(a, b, c) + f_x(a, b, c) \cdot h + f_y(a, b, c) \cdot k + f_z(a, b, c) \cdot l$
$= f(a, b, c) + \left(f_x(a, b, c), f_y(a, b, c), f_z(a, b, c)\right) \cdot (h, k, l)$

We define
$df = f_x(x, y, z) \cdot dx + f_y(x, y, z) \cdot dy + f_z(x, y, z) \cdot dz$

Example:
We have constructed a metal cube that is supposed to have edge length $100\ mm$, but each of its three measured dimensions x, y and z may be in error by as much as a millimeter. Use differentials to estimate the maximum resulting error in its calculated volume $V = xyz$.

Solutions
$V(x, y, z) = xyz$
$V_x(x, y, z) = yz$; $V_y(x, y, z) = xz$; $V_z(x, y, z) = xy$
$dV = V_x(x, y, z)dx + V_y(x, y, z)dy + V_z(x, y, z)dz = yzdx + xzdy + xydz$
$\Delta V \approx dV = 100 \times 100 \times \pm 1 + 100 \times 100 \times \pm 1 + 100 \times 100 \times \pm 1$
Note: $100 \times 100 \times 1 + 100 \times 100 \times 1 + 100 \times 100 \times 1 = 30000$
the maximum resulting error in its calculated volume
$\approx \pm 30000$ (in mm^3)

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

<u>**n – Dimensional Case:**</u>
Let $f: R^n \to R$ be a function and $A, H \in R^n$.
Suppose $f_{x_1}, f_{x_2}, \cdots, f_{x_n}$ are continuous at points near to A.
We can show that $f(A+H) \approx f(A) + \left(f_{x_1}(A), f_{x_2}(A), \cdots, f_{x_n}(A)\right) \cdot H$ when $\|H\| \approx 0$.
We define the gradient of f at A as $grad\ f(A) = \nabla f(A) = \left(f_{x_1}(A), f_{x_2}(A), \cdots, f_{x_n}(A)\right)$.
Then, $f(A+H) \approx f(A) + \nabla f(A) \cdot H$ when $\|H\| \approx 0$.

At general point $X = (x_1, x_2, \cdots, x_n)$, we define the gradient of f at X as
$grad\ f(X) = \nabla f(X) = \left(f_{x_1}(X), f_{x_2}(X), \cdots, f_{x_n}(X)\right)$.

Then, $f(X+H) \approx f(X) + \nabla f(X) \cdot H$ when $\|H\| \approx 0$.

<u>**Linear Approximation**</u>
Let $f: R^n \to R$ be a function and $A, H \in R^n$.
Suppose we can find:
(i) $\delta \in R$ with $\delta > 0$ AND
(ii) $\nabla f(A)$
so that there exists a function $\boldsymbol{\varepsilon}: B(O,\delta) \to R$ such that $\boldsymbol{\varepsilon}(H) \to 0$ as $\|H\| \to 0$
<u>**AND**</u> $f(A+H) = f(A) + \nabla f(A) \cdot H + \boldsymbol{\varepsilon}(H) \cdot \|H\|$.
In this case, we say $f(A) + \nabla f(A) \cdot H$ is <u>**the linear approximation**</u> of $f(A+H)$ when $\|H\| \approx 0$.

<u>**Concept of Differentiability**</u>
$\frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|} = \frac{\boldsymbol{\varepsilon}(H) \cdot \|H\|}{\|H\|} = \boldsymbol{\varepsilon}(H) \to 0$ as $\|H\| \to 0$
So, $\lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|} = 0$

<u>**Remark 1:**</u>
The property "the linear approximation" $\Longrightarrow \lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|} = 0$

<u>**Remark 2:**</u>
Suppose $\lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|} = 0$.
We may define $\boldsymbol{\varepsilon}: R^n \to R$ by $\boldsymbol{\varepsilon}(H) = \frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|}$.
Then, $f(A+H) = f(A) + \nabla f(A) \cdot H + \boldsymbol{\varepsilon}(H) \cdot \|H\|$ and $\boldsymbol{\varepsilon}(H) \to 0$ as $\|H\| \to 0$

$\lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|} = 0 \Longrightarrow$ The property "the linear approximation"

<u>**Definition**</u>
Let $f: R^n \to R$ be a function and $A \in R^n$.
Suppose we can find:
(i) $\delta \in R$ with $\delta > 0$ AND
(ii) $\nabla f(A)$
so that there exists a function $\boldsymbol{\varepsilon}: B(O,\delta) \to R$ such that $\boldsymbol{\varepsilon}(H) \to 0$ as $\|H\| \to 0$
<u>**AND**</u> $f(A+H) = f(A) + \nabla f(A) \cdot H + \boldsymbol{\varepsilon}(H) \cdot \|H\|$.
In this case, we say <u>**f is differentiable at A**</u>.

Remark: $\lim_{\|H\| \to 0} \frac{f(A+H) - f(A) - \nabla f(A) \cdot H}{\|H\|} = 0$.
<u>**Definition**</u>
Let $f: R^n \to R$ be a function and $A \in R^n$. We say f is <u>**continuously differentiable at A**</u> if we can find $r \in R$ with $r > 0$ such that $f_{x_1}, f_{x_2}, \cdots, f_{x_n}$ are continuous on $B(A, r)$.

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Theorem:
Let $f: R^n \to R$ be a function and $A \in R^n$.
f is **continuously differentiable at A** $\Longrightarrow$ **f is differentiable at A**

Remarks:
(i) the converse in general is not true
(ii) **f is differentiable at A** $\Longrightarrow$
we can find $\nabla f(A)$, but $f_{x_1}, f_{x_2}, \cdots, f_{x_n}$ may not be continuous near A

Theorem:
Let $f: R^n \to R$ be a function and $A \in R^n$.
f is **differentiable at A** $\Longrightarrow$ **f is continuous at A**

Definition
Let $f: R^n \to R$ be a function.
Let $\phi \neq S \subset R^n$ and S is an open set.
We say f is **differentiable on S** if f is differentiable at A for any $A \in S$.

Example 1
Let $f: R^2 \to R$ be defined by $f(x, y) = xy$.
Show that f is differentiable at $(1,2)$.

Proof
$f(1,2) = 2$; $f_x(x, y) = y$; $f_x(1,2) = 2$; $f_y(x, y) = x$; $f_y(1,2) = 1$;
$\nabla f(1,2) \cdot (h, k) = (2,1) \cdot (h, k) = 2h + k$
$f(1 + h, 2 + k) = (1 + h)(2 + k) = 2 + 2h + k + hk = f(1,2) + \nabla f(1,2) \cdot (h, k) + hk$

Let $\varepsilon: R^2 \to R$ be defined by $\varepsilon(h, k) = \begin{cases} \frac{hk}{\sqrt{h^2+k^2}} & if (h, k) \neq (0,0) \\ 0 & if (h, k) = (0,0) \end{cases}$.
Note: $\|(h, k)\| = \sqrt{h^2 + k^2}$.
Then, $f(1 + h, 2 + k) = f(1,2) + \nabla f(1,2) \cdot (h, k) + \varepsilon(h, k)\|(h, k)\|$.

$$\lim_{\|(h,k)\|\to 0} \varepsilon(h, k) = \lim_{\|(h,k)\|\to 0} \frac{hk}{\sqrt{h^2 + k^2}} = \lim_{r\to 0^+} \frac{1}{2} r sin2\theta = 0$$

Reason:
Let $h = rcos\theta$, $k = rsin\theta$ where $r \geq 0$.
Then, $\frac{hk}{\sqrt{h^2+k^2}} = \frac{r^2 sin\theta cos\theta}{r} = \frac{1}{2} r sin2\theta$
Note: $|sin2\theta| \leq 1$

Thus, f is differentiable at $(1,2)$.

Example 2

Let $f: R^2 \to R$ be defined by $f(x,y) = \sqrt{x^2 + y^2}$.
Show that f is not differentiable at $(0,0)$.

Proof:

Suffices to show $f_x(0,0)$ doesn't exist.
$f(0,0) = 0$. $f(0+h,0) = f(h,0) = \sqrt{h^2 + 0} = |h|$. $f(0+h,0) - f(0,0) = |h|$.

$$\lim_{h\to 0^+} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

$$\lim_{h\to 0^-} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1$$

$$\lim_{h\to 0^+} \frac{f(0+h,0) - f(0,0)}{h} = 1 \neq -1 = \lim_{h\to 0^-} \frac{f(0+h,0) - f(0,0)}{h}$$

So, $\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$ doesn't exist.
Thus, $f_x(0,0)$ doesn't exist.

Rules for Differentiation

Theorem:

Let $\phi \neq S \subset R^n$ and S is an open set.
Let $f: S \to R$ and $g: S \to R$ be functions.
Let $\lambda \in R$ and $P \in S$.
Suppose both f and g are differentiable at P.
Then,

(i) $f + g$ is differentiable at P
(ii) $f - g$ is differentiable at P
(iii) $f \cdot g$ is differentiable at P
(iv) $\frac{f}{g}$ is differentiable at P
(Assumed that we can find $r \in R$ with $r > 0$ such that $B(P,r) \subset S$ and $g(X) \neq 0$ for any $X \in B(P,r)$.)
(v) λf is differentiable at P

Proof: Omitted (As Exercises)

 # Multivariable Chain Rule

Theorem 1:

Let $x: I \to R$ and $y: I \to R$ are functions, where I is an open interval.
Suppose x and y are differentiable on I.
Let $\phi \neq S \subset R^2$ and S is an open set.
Suppose $\{(x(t), y(t)): t \in I\} \subset S$.
Let $f: S \to R$ be a function.
Suppose all partial derivatives of f are continuous on S.
Then we can define a function $z: I \to R$ by $z(t) = f(x(t), y(t))$ and it is differentiable on I.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Remark: Sometimes, if we write $w = f(x,y)$, then we also write $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$
by considering $w = w(t) = f(x(t), y(t))$.

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Idea of the proof

$$\Delta z \approx dz = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial f}{\partial x}\cdot\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\cdot\frac{\Delta y}{\Delta t}$$

Taking the limit $\Delta t \to 0$, we get $\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$.

Example 1:

Suppose that $w = e^{xy}$, $x = t^2$ and $y = t^3$. Find $\frac{dw}{dt}$.

Solutions

Method 1

$w = e^{xy} = e^{(t^2\cdot t^3)} = e^{(t^5)}$

$\frac{dw}{dt} = e^{(t^5)}\cdot\frac{d}{dt}t^5 = 5t^4\cdot e^{(t^5)}$

Method 2 (By Chain Rule)

$\frac{\partial w}{\partial x} = ye^{xy}; \frac{dx}{dt} = 2t; \frac{\partial w}{\partial y} = xe^{xy}; \frac{dy}{dt} = 3t^2$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$

$= ye^{xy}\cdot 2t + xe^{xy}\cdot 3t^2$

$= 2t^4\cdot e^{(t^5)} + 3t^4\cdot e^{(t^5)}$

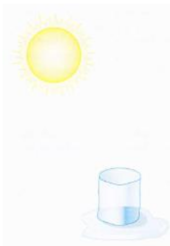
$= 5t^4\cdot e^{(t^5)}$

Example 2:

The figure shows a melting cylindrical block of ice.

Because of the sun's heat beating down from above, its height h is decreasing more rapidly than its radius r.

If its height is decreasing at $3\ cm/h$ and its radius is decreasing at $1\ cm/h$ when $r = 15\ cm$ and $h = 40\ cm$, what is the rate of change of the volume V of the block at that instant?



Solutions

As $V = \pi r^2 h$, by Chain Rule, $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$.

When $r = 15$ and $h = 40$, $\frac{dh}{dt} = -3$, $\frac{dr}{dt} = -1$ (minus sign means decreasing).

$\frac{dV}{dt} = 2\pi\cdot 15\cdot 40\cdot(-1) + \pi\cdot 15^2\cdot(-3) = -1875\pi \approx -5890.49$ (in cm^3/h).

The volume of the block at that instant is decreasing at the rate of 5890 cm^3/h.

Example 3:

Find $\frac{dw}{dt}$ if $w = x^2 + ze^y + sinxz, x = t, y = t^2, z = t^3$.

Solutions

Method 1

$w = x^2 + ze^y + sinxz$

$= t^2 + t^3e^{(t^2)} + sin(t^4)$

$\frac{dw}{dt}$

$= 2t + 3t^2e^{(t^2)} + t^3e^{(t^2)}\cdot 2t + cos(t^4)\cdot 4t^3$

$= 2t + (3t^2 + 2t^4)e^{(t^2)} + 4t^3cos(t^4)$

Method 2 (By Chain Rule)

$\frac{\partial w}{\partial x} = 2x + zcosxz; \frac{dx}{dt} = 1$

$\frac{\partial w}{\partial y} = ze^y; \frac{dy}{dt} = 2t$

$\frac{\partial w}{\partial z} = e^y + xcosxz; \frac{dz}{dt} = 3t^2$

$\frac{dw}{dt}$

$= (2x + zcosxz)\cdot 1 + (ze^y)\cdot 2t + (e^y + xcosxz)\cdot 3t^2$

$= 2t + t^3cos(t^4) + 2t^4e^{(t^2)} + 3t^2e^{(t^2)} + 3t^3cos(t^4)$

$= 2t + (3t^2 + 2t^4)e^{(t^2)} + 4t^3cos(t^4)$

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

Theorem 2:

Let $x_i: I \to R$ is a function, for $i = 1,2,\cdots,n$, where I is an open interval.
Suppose x_i is differentiable on I, for $i = 1,2,\cdots,n$.
Let $\phi \neq S \subset R^n$ and S is an open set.
Suppose $\{(x_1(t), x_2(t), \cdots, x_n(t)): t \in I\} \subset S$.
Let $f: S \to R$ be a function.
Suppose all partial derivatives of f are continuous on S.
Then we can define a function $z: I \to R$ by $z(t) = f(x_1(t), x_2(t), \cdots, x_n(t))$ and it is differentiable on I.

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{dt}$$

Theorem 3 (General Chain Rule):

Let $\phi \neq T \subset R^n$ and T is an open set.
Let $x_i: T \to R$ is a function, for $i = 1,2,\cdots,m$.
Suppose all partial derivatives of x_i are continuous on T, for $i = 1,2,\cdots,m$.
Let $\phi \neq S \subset R^m$ and S is an open set.
Suppose $\{(x_1(A), x_2(A), \cdots, x_m(A)): A \in T\} \subset S$.
Let $f: S \to R$ be a function.
Suppose all partial derivatives of f are continuous on S.
Then we can define a function $z: T \to R$ by $z(A) = f(x_1(A), x_2(A), \cdots, x_m(A))$ and all its partial derivatives are continuous on T.

$$\frac{\partial z}{\partial t_k} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t_k}$$

Example 4:

Suppose $z = f(u,v)$, $u = 2x + y$, $v = 3x - 2y$.
Given the values of $\frac{\partial z}{\partial u} = 3$ and $\frac{\partial z}{\partial v} = -2$ at the point $(u,v) = (3,1)$.
Find the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the corresponding point $(x,y) = (1,1)$.

Solutions

At $(x,y) = (1,1)$ and $(u,v) = (3,1)$,

$\frac{\partial u}{\partial x} = 2; \frac{\partial v}{\partial x} = 3$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = 3 \times 2 + (-2) \times 3 = 0$$

$\frac{\partial u}{\partial y} = 1; \frac{\partial v}{\partial y} = -2$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = 3 \times 1 + (-2) \times (-2) = 7$$

Example 5:

Let $w = f(x,y)$ where x and y are given in polar coordinates by the equations $x = rcos\theta$ and $y = rsin\theta$.
Calculate $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial \theta}$ and $\frac{\partial^2 w}{\partial r^2}$ in terms of r, θ and the partial derivatives of w with respect to x and y.

Solutions

$\frac{\partial x}{\partial r} = cos\theta; \frac{\partial x}{\partial \theta} = -rsin\theta; \frac{\partial y}{\partial r} = sin\theta; \frac{\partial y}{\partial \theta} = rcos\theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} = cos\theta \frac{\partial w}{\partial x} + sin\theta \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -rsin\theta \frac{\partial w}{\partial x} + rcos\theta \frac{\partial w}{\partial y}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{\partial}{\partial r}\left(cos\theta \frac{\partial w}{\partial x} + sin\theta \frac{\partial w}{\partial y}\right)$$

$$= cos\theta \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial x}\right) + sin\theta \frac{\partial}{\partial r}\left(\frac{\partial w}{\partial y}\right)$$

$$= cos\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) \cdot \frac{\partial y}{\partial r}\right) + sin\theta\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) \cdot \frac{\partial y}{\partial r}\right)$$

$$= cos\theta\left(\frac{\partial^2 w}{\partial x^2} \cdot cos\theta + \frac{\partial^2 w}{\partial y \partial x} \cdot sin\theta\right) + sin\theta\left(\frac{\partial^2 w}{\partial x \partial y} \cdot cos\theta + \frac{\partial^2 w}{\partial y^2} \cdot sin\theta\right)$$

$$= cos^2\theta \frac{\partial^2 w}{\partial x^2} + 2sin\theta cos\theta \frac{\partial^2 w}{\partial y \partial x} + sin^2\theta \frac{\partial^2 w}{\partial y^2}$$

Note: $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

Example 6:

Suppose that $w = f(u,v,x,y)$ where u and v are functions of x and y.
Find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.
[Hint: x and y play dual roles as intermediate and independent variables.]

Solutions

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial y}$$

Example 7:

Consider a parametric curve $x = x(t), y = y(t), z = z(t)$ that lies on the surface $z = f(x,y)$ in space.
Recall that if $\vec{T} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ and $\vec{N} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$, then $\vec{T}$ is tangent to the curve and $\vec{N}$ is normal to the surface.
Show that $\vec{T}$ and $\vec{N}$ are everywhere perprndicular.

Proof:

$$\vec{T} \cdot \vec{N} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \cdot \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

$$= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} - \frac{dz}{dt}$$

$$= \frac{dz}{dt} - \frac{dz}{dt} = 0$$

So, $\vec{T}$ and $\vec{N}$ are everywhere perprndicular.

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Exercise

Suppose $f(x,y)$ satisfy $f(tx,ty)=t^m f(x,y)$ for any $(x,y)\in R^2$, where m is a fixed positive integer.
Show that $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=mf$.
[Hint: Consider $\frac{\partial}{\partial t}f(tx,ty)$.]

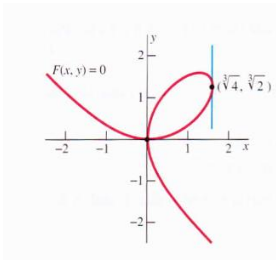
 # Implicit Partial Differentiation

Theorem:

Suppose that the function $F(x_1,x_2,\cdots,x_n,z)$ is continuously differentiable near to the point $(a_1,a_2,\cdots,a_n,b)$ at which $F(a_1,a_2,\cdots,a_n,b)=0$ and $\frac{\partial F}{\partial z}\neq 0$.
Then, there exists a continuously differentiable function $z=g(x_1,x_2,\cdots,x_n)$ such that $b=g(a_1,a_2,\cdots,a_n)$ and $F\big(x_1,x_2,\cdots,x_n,g(x_1,x_2,\cdots,x_n)\big)=0$ for $(x_1,x_2,\cdots,x_n)$ near $(a_1,a_2,\cdots,a_n)$.

Example 1:

Consider the graph of the equation $F(x,y)=x^3+y^3-3xy=0$,
find $\frac{dy}{dx}$ if it is well defined.



Solutions

Note: $F(x,y)=x^3+y^3-3xy$

$$0=\frac{d}{dx}F(x,y)=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=3x^2-3y+(3y^2-3x)\frac{dy}{dx}$$

So, $\frac{dy}{dx}=-\frac{3x^2-3y}{3y^2-3x}=-\frac{x^2-y}{y^2-x}$ (Assumed $y^2-x\neq 0$)
Consider $y^2-x=0$ and $x^3+y^3-3xy=0$, we have
$y^6+y^3-3y^3=0\Longrightarrow y^6-2y^3=0\Longrightarrow y^3(y^3-2)=0\Longrightarrow y=0$ or $\sqrt[3]{2}$
When $y=0, x=0$.
When $y=\sqrt[3]{2}, x=\sqrt[3]{4}$.
$\frac{dy}{dx}$ is undefined at points $(0,0)$ and $(\sqrt[3]{4},\sqrt[3]{2})$.

Example 2:

Suppose $w=G(x,y)$, $u=u(x,y)$ and $v=v(x,y)$ be given.
Suppose we know that x and y can be solved in terms of u and v.
Find $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ in terms of $\frac{\partial w}{\partial u},\frac{\partial w}{\partial v},\frac{\partial x}{\partial u},\frac{\partial x}{\partial v},\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$.

Solutions

$$\frac{\partial w}{\partial u}=\frac{\partial G}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial G}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v}=\frac{\partial G}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial G}{\partial y}\cdot\frac{\partial y}{\partial v}$$

In Matrix Form

$$\begin{pmatrix}\frac{\partial w}{\partial u}\\ \frac{\partial w}{\partial v}\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial G}{\partial x}\\ \frac{\partial G}{\partial y}\end{pmatrix},\text{ so }\begin{pmatrix}\frac{\partial G}{\partial x}\\ \frac{\partial G}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{pmatrix}^{-1}\begin{pmatrix}\frac{\partial w}{\partial u}\\ \frac{\partial w}{\partial v}\end{pmatrix}=\frac{1}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}\begin{pmatrix}\frac{\partial y}{\partial v} & -\frac{\partial y}{\partial u}\\ -\frac{\partial x}{\partial v} & \frac{\partial x}{\partial u}\end{pmatrix}\begin{pmatrix}\frac{\partial w}{\partial u}\\ \frac{\partial w}{\partial v}\end{pmatrix}$$

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Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

<u># Directional Derivatives and Gradient Vector</u>

<u>Concept of partial derivative</u>

Suppose $n = 2,3,\cdots$.
Let $\phi \neq S \subset R^n$ and S is an open set.
Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$.
Let $E_j = (e_{j1}, e_{j2}, \cdots, e_{jn})$ be defined by $e_{ji} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$.
That is, only $j-$ th cordinate is 1, other coordinates are zeros.
Note: E_j is an unit vector in the direction of the coordinate axis for x_j.
$f_{x_j}(x_1, x_2, \cdots, x_n) = \frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(X + hE_j) - f(X)}{h}$.

<u>Concept of directional derivative</u>

Suppose $n = 2,3,\cdots$.
Let $\phi \neq S \subset R^n$ and S is an open set.
Let $f: S \to R$ be a function on $X = (x_1, x_2, \cdots, x_n)$.
Let $\boldsymbol{u}$ be any unit vector.
We define:
$D_{\boldsymbol{u}} f(x_1, x_2, \cdots, x_n) = \lim_{h \to 0} \frac{f(X + h\boldsymbol{u}) - f(X)}{h}$.

<u>Theorem:</u>

$\boldsymbol{D_u f(X) = \nabla f(X) \cdot u}$

Proof:
$D_{\boldsymbol{u}} f(X)$
$$= \lim_{h \to 0} \frac{f(X + h\boldsymbol{u}) - f(X)}{h} = \lim_{h \to 0} \frac{\nabla f(X) \cdot (h\boldsymbol{u})}{h} = \lim_{h \to 0} \frac{h \nabla f(X) \cdot \boldsymbol{u}}{h} = \lim_{h \to 0} \nabla f(X) \cdot \boldsymbol{u} = \nabla f(X) \cdot \boldsymbol{u}$$

<u>Example:</u>

Suppose $f(x, y) = \frac{1}{180}(7400 - 4x - 9y - 0.03xy)$ for any $(x, y) \in R^2$.
Find $D_{\boldsymbol{u}} f\big((200,200)\big)$ where $\boldsymbol{u}$ is the unit vector in the direction of $\boldsymbol{v} = (3,4)$.

<u>Solutions</u>

$f_x(x, y) = \frac{1}{180}(-4 - 0.03y)$, $f_x(200,200) = \frac{-1}{18}$;
$f_y(x, y) = \frac{1}{180}(-9 - 0.03x)$, $f_y(200,200) = \frac{-1}{12}$;
$\nabla f\big((200,200)\big) = \left(\frac{-1}{18}, \frac{-1}{12}\right)$.
$\|\boldsymbol{v}\| = \|(3,4)\| = \sqrt{3^2 + 4^2} = 5$
$$\boldsymbol{u} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} = \frac{1}{5}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$
$$D_{\boldsymbol{u}} f\big((200,200)\big) = \nabla f\big((200,200)\big) \cdot \boldsymbol{u} = \left(\frac{-1}{18}, \frac{-1}{12}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{-1}{10}$$

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Application:

If $f(x, y)$ denotes the temperature (in degrees Celsius) at the point (x, y) near an airport where distances x and y are measured in kilometers, then $D_{\boldsymbol{u}}f\big((200,200)\big)$ will be the initial rate of change of temperature when the aircraft heads northeast in the direction specified by the vector $\boldsymbol{v}$ at the location $(200,200)$.

Note:
$D_{\boldsymbol{u}}f\big((200,200)\big) = -0.1$ means "The instantaneous rate of change is decreasing at $0.1\ {}^{O}C/km$".

Significance of the Gradient Vector

Suppose θ is the angle between $\nabla f(X)$ and $\boldsymbol{u}$.
$\boldsymbol{D_u f(X) = \nabla f(X) \cdot u = \|\nabla f(X)\| \cdot \|u\| cos\theta = \|\nabla f(X)\| cos\theta}$

Note:
The maximum value of $D_u f(X)$ is $\|\nabla f(X)\|$.
The maximum value is obtained when $cos\theta = 1$, that is $\boldsymbol{u}$ is in the same direction as $\nabla f(X)$.
In this case, $\boldsymbol{u} = \frac{1}{\|\nabla f(X)\|}\nabla f(X)$.

Geometric Meaning of the Gradient Vector

Suppose $\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big)$ is a curve on the surface $F(x, y, z) = 0$ where F is continuously differentiable.

$$0 = F\big(x(t), y(t), z(t)\big)$$
$$0 = \frac{d}{dt}0 = \frac{d}{dt}F\big(x(t), y(t), z(t)\big) = \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt}$$
$$= \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \nabla F \cdot \boldsymbol{r}'$$

∇F is always perpendicular to the tangent vector of the curve on the surface.
So, ∇F is a normal vector of the tangent plane.

Application:

Suppose $F(x, y, z) = z - f(x, y)$.
$\nabla F = \big(-f_x(x, y, z), -f_y(x, y, z), 1\big)$ is a normal vector of the surface $z = f(x, y)$.

Example:

Write an equation of the plane tangent to the ellipsoid $2x^2 + 4y^2 + z^2 = 45$ at the point $P(2, -3, -1)$.

Solutions

Let $F(x, y, z) = 2x^2 + 4y^2 + z^2 - 45$ for any $(x, y, z) \in R^3$.
$\nabla F(x, y, z) = (4x, 8y, 2z)$
$\nabla F(2, -3, -1) = (8, -24, -2)$ is a normal vector of required tangent plane.
An equation of required tangent plane is
$\big((x, y, z) - (2, -3, -1)\big) \cdot (8, -24, -2) = 0$
$8x - 24y - 2z - (16 + 72 + 2) = 0$
$8x - 24y - 2z - 90 = 0$
$4x - 12y - z - 45 = 0$

Theorem:

Suppose F and G are continuously differentiable. The intersection of $F(x, y, z) = 0$ and $G(x, y, z) = 0$ will be some sort of curve in space.

If $P(a, b, c)$ is a point of such curve such that $\nabla F(P)$ and $\nabla G(P)$ are not collinear, then $\nabla F(P) \times \nabla G(P)$ will be a vector parallelt to the tangent vector of the curve (the intersection of the two surfaces) at P.

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<u>Example 1:</u>

The point $P(1,-1,2)$ lies on both paraboloids $F(x,y,z) = x^2 + y^2 - z = 0$ and $G(x,y,z) = 2x^2 + 3y^2 + z^2 - 9 = 0$.

Write an equation of the plane through P and is normal to the curve of intersection of these two surfaces.

<u>Solutions</u>

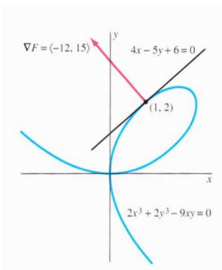
$F(x,y,z) = x^2 + y^2 - z$
$\nabla F(x,y,z) = (2x, 2y, -1); \nabla F(P) = \nabla F(1,-1,2) = (2,-2,-1)$
$G(x,y,z) = 2x^2 + 3y^2 + z^2 - 9$
$\nabla G(x,y,z) = (4x, 6y, 2z); \nabla G(P) = \nabla G(1,-1,2) = (4,-6,4)$
$\nabla F(P) \times \nabla G(P)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & -1 \\ 4 & -6 & 4 \end{vmatrix} = -14\vec{i} - 12\vec{j} - 4\vec{k} = (-14,-12,-4)$$

An equation of required tangent plane is
$\big((x,y,z) - (1,-1,2)\big) \cdot (-14,-12,-4) = 0$
$-14x - 12y - 4z - (-14 + 12 - 8) = 0$
$-14x - 12y - 4z + 10 = 0$
$7x + 6y + 2z - 5 = 0$

<u>Example 2:</u>

Write an equation of the line tangent at the point $P(1,2)$ to the folium of Descartes with equation $F(x,y) = 2x^3 + 2y^3 - 9xy = 0$.



<u>Solutions</u>

$F(x,y) = 2x^3 + 2y^3 - 9xy$
$\nabla F(x,y) = (6x^2 - 9y, 6y^2 - 9x); \nabla F(P) = \nabla F(1,2) = (-12,15)$
A vector normal to required tangent line is $(-12,15)$.
For any point (x,y) on required tangent line, $(x,y) - (1,2)$ is a vector in the direction of the required tangent line.
An equation of required tangent line is $\big((x,y) - (1,2)\big) \cdot (-12,15) = 0$.
$-12(x-1) + 15(y-2) = 0$
$-12x + 12 + 15y - 30 = 0$
$-12x + 15y - 18 = 0$
$4x - 5y + 6 = 0$

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 # Lagrange Multiplers and Constrained Optimization
Theorem (Two Dimensional Case)

Let $f(x,y)$ and $g(x,y)$ be continuously differentiable functions.

If the maximum value (or minimum value) of $f(x,y)$ subject to the constraint $g(x,y)=0$ occur at a point $P(x_0,y_0)$ where $\nabla g(P)\neq(0,0)$, then $\nabla f(P)=\lambda\nabla g(P)$ for some constant λ.

Proof for the case (maximum value at $P(x_0,y_0)$)

Suppose the maximum value of $f(x,y)$ subject to the constraint $g(x,y)=0$ occurs at a point $P(x_0,y_0)$ where $\nabla g(P)\neq(0,0)$.

We consider a curve on $g(x,y)=0$ and passing through P, say $r:(-1,1)\to R^2$,
$r(t)=\big(x(t),y(t)\big)$ and $r(0)=P(x_0,y_0)$.
$0=\frac{d}{dt}f\big(x(t),y(t)\big)\Big|_{t=0}$ (as P is a local maxima on $g(x,y)=0$)
$$0=\frac{d}{dt}f\big(x(t),y(t)\big)\Big|_{t=0}=\nabla f\big(x(t),y(t)\big)\cdot r'(t)\big|_{t=0}=\nabla f(P)\cdot r'(0)$$
This is true for every curve on $g(x,y)=0$ and passing through P.
So, $\nabla f(P)$ is normal to any tangent vector of every curve that is on $g(x,y)=0$ and is passing through P.

Also, $0=g\big(x(t),y(t)\big)$. We have $0=\frac{d}{dt}g\big(x(t),y(t)\big)=\frac{d}{dt}g\big(x(t),y(t)\big)\Big|_{t=0}$
$=\nabla g\big(x(t),y(t)\big)\cdot r'(t)\big|_{t=0}=\nabla g(P)\cdot r'(0)$
This is true for every curve on $g(x,y)=0$ and passing through P.
So, $\nabla g(P)$ is normal to any tangent vector of every curve that is on $g(x,y)=0$ and is passing through P.
As $\nabla g(P)\neq(0,0)$, $\nabla f(P)$ and $\nabla g(P)$ must be parallel to each other.
So, $\nabla f(P)=\lambda\nabla g(P)$ for some constant λ.

Remark: We may generalize to $\mathbf{n}$ −dimensional case.

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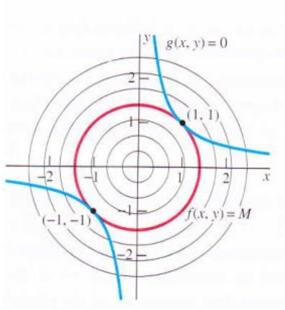
Example 1:

Find the points of the rectangular hyperbola $xy = 1$ that are closest to the origin $(0,0)$.



Solutions

Let $d(x,y) = \sqrt{x^2 + y^2}$ for any $(x,y) \in R^2$.
Let $f(x,y) = x^2 + y^2$ for any $(x,y) \in R^2$.
Let $g(x,y) = xy - 1$ for any $(x,y) \in R^2$.
$d(x_0, y_0)$ is a solution for "Minimize $d(x,y)$ subject to $g(x,y) = 0$" $\Leftrightarrow$
$f(x_0, y_0)$ is a solution for "Minimize $f(x,y)$ subject to $g(x,y) = 0$"

Consider the problem "Minimize $f(x,y) = x^2 + y^2$ subject to $g(x,y) = 0$",
$\nabla f(x,y) = (2x, 2y)$, $\nabla g(x,y) = (y, x)$
Put $\nabla f(x,y) = \lambda \nabla g(x,y)$, we have $(2x, 2y) = \lambda(y, x)$

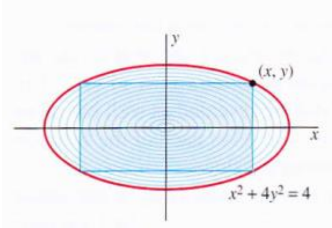
$$\begin{cases} 2x = \lambda y \\ 2y = \lambda x \end{cases}$$

So, $4y = \lambda(2x) = \lambda(\lambda y) = \lambda^2 y \Longrightarrow (\lambda - 2)(\lambda + 2)y = 0 \Longrightarrow \lambda = 2$ or -2 or $y = 0$
$y = 0$ must be rejected as $xy = 1$
For $\lambda = 2$, $x = y$, so $x^2 = 1$ (as $xy = 1$), $x = 1$ or -1. The two points are $(1,1)$ and $(-1,-1)$.
For $\lambda = -2$, $x = -y$, so $-y^2 = 1$ (as $xy = 1$), $y^2 = -1$. No real solutions.

Thus, the two points are $(1,1)$ and $(-1,-1)$.

Example 2:

What is the maximal cross-sectional area of a rectangular beam cut as indicated from an elliptical log with semi-axes of lengths $a = 2\ ft.$ and $b = 1\ ft.$?



Solutions

An equation of the given ellipse is $\frac{x^2}{2^2} + y^2 = 1$.
Let $A(x, y) = 4xy$ for any $(x, y) \in R^2$.
Let $g(x, y) = \frac{1}{4}x^2 + y^2 - 1$ for any $(x, y) \in R^2$.
We consider "Maximize $A(x, y)$ subject to $g(x, y) = 0$".
$\nabla A(x, y) = (4y, 4x)$, $\nabla g(x, y) = \left(\frac{1}{2}x, 2y\right)$
Put $\nabla A(x, y) = \lambda \nabla g(x, y)$, we have $(4y, 4x) = \lambda\left(\frac{1}{2}x, 2y\right)$

$$\begin{cases} 4y = \frac{1}{2}\lambda x \\ 4x = 2\lambda y \end{cases}$$

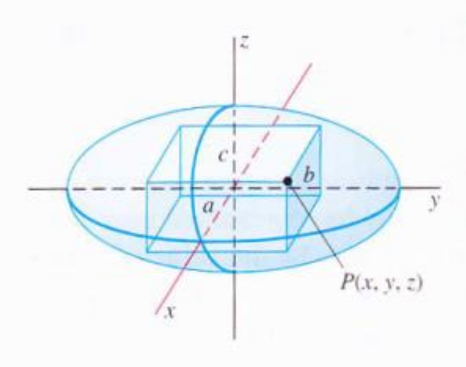
$8x = \lambda(4y) = \lambda\left(\frac{1}{2}\lambda x\right) = \frac{1}{2}\lambda^2 x \Rightarrow 16x = \lambda^2 x \Rightarrow (\lambda - 4)(\lambda + 4)x = 0 \Rightarrow \lambda = 4$ or -4 or $x = 0$
But $x = 0$ must be rejected, otherwise $x = 0 = y$ (But it doesn't satisfy $\frac{1}{4}x^2 + y^2 = 1$)
For $\lambda = 4$, $4y = 2x$, $x = 2y$. Also, we have $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow y^2 + y^2 = 1 \Rightarrow 2y^2 = 1 \Rightarrow y = \frac{\pm 1}{\sqrt{2}}$.
The four points on the ellipse for this case are $\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{2}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{-2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-2}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
For $\lambda = -4$, $4y = -2x$, $x = -2y$.
Also, we have $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow y^2 + y^2 = 1 \Rightarrow 2y^2 = 1 \Rightarrow y = \frac{\pm 1}{\sqrt{2}}$.
The four points on the ellipse for this case are $\left(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{2}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{-2}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-2}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.
The maximal cross-sectional area is $4 \times \frac{2}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 4$ (in ft^2.)

Remark:
Area of the ellipse is $\pi ab = 2\pi$.

$$\frac{4}{2\pi} \times 100\% \approx 63.66\%$$

Example 3:

Find the maximum volume of a rectangular box inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with its faces parallel to the coordinate planes.
(Assumed $a>0$, $b>0$ and $c>0$.)



Solutions

Let $V(x,y,y)=8xyz$ for any $(x,y,z)\in R^3$.
Let $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ for any $(x,y,z)\in R^3$.
We consider "Maximize $V(x,y,z)$ subject to $g(x,y,z)=0$".
$\nabla V(x,y,z)=(8yz,8xz,8xy)$, $\nabla g(x,y,z)=\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right)$
Put $\nabla V(x,y,z)=\lambda\nabla g(x,y,z)$, we have $(8yz,8xz,8xy)=\lambda\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right)$

$$\begin{cases}8yz=\dfrac{2\lambda x}{a^2}\\ 8xz=\dfrac{2\lambda y}{b^2}\\ 8xy=\dfrac{2\lambda z}{c^2}\end{cases}$$

$$\frac{2\lambda x^2}{a^2}=\frac{2\lambda y^2}{b^2}=\frac{2\lambda z^2}{c^2}=8xyz\Longrightarrow\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

Also, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Thus, $\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}$.
Assume $x>0$, $y>0$ and $z>0$, we have $x=\frac{1}{\sqrt{3}}a$, $y=\frac{1}{\sqrt{3}}b$ and $z=\frac{1}{\sqrt{3}}c$.
The maximum volume is $8\times\frac{1}{\sqrt{3}}a\times\frac{1}{\sqrt{3}}b\times\frac{1}{\sqrt{3}}c=\frac{8\sqrt{3}}{9}abc$.
Remark:
The volume of the ellipsoid is $\frac{4}{3}\pi abc$.

$$\frac{\frac{8\sqrt{3}}{9}abc}{\frac{4}{3}\pi abc}\times100\%=\frac{2\sqrt{3}}{3\pi}\times100\%\approx36.76\%$$

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

 # With 2 Constraints

Theorem (Three Dimensional Case)

Let $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ be continuously differentiable functions.

If the maximum value (or minimum value) of $f(x, y, z)$ subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$ occur at a point $P(x_0, y_0, z_0)$ where $\nabla g(P) \neq (0,0,0)$ and $\nabla h(P) \neq (0,0,0)$, then
$\nabla f(P) = \lambda_1 \nabla g(P) + \lambda_2 \nabla h(P)$ for some constants λ_1 and λ_2.

Proof for the case (maximum value at $P(x_0, y_0, z_0)$)

Suppose the maximum value of $f(x, y, z)$ subject to the constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$ occurs at a point $P(x_0, y_0, z_0)$ where $\nabla g(P) \neq (0,0,0)$ and $\nabla h(P) \neq (0,0,0)$.

We consider a curve that is on the intersection of $g(x, y, z) = 0$ and $h(x, y, z) = 0$ and is passing through P, say $r: (-1,1) \to R^3$, $r(t) = \big(x(t), y(t), z(t)\big)$ and $r(0) = P(x_0, y_0, z_0)$.

Similar to the proof for one constraint case, we have

$\nabla f(P) \cdot r'(0) = 0$
$\nabla g(P) \cdot r'(0) = 0$
$\nabla h(P) \cdot r'(0) = 0$

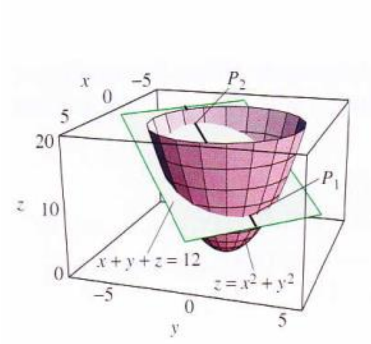
As $\nabla g(P) \neq (0,0,0)$ and $\nabla h(P) \neq (0,0,0)$, they are also non-parallel,
$f(P)$ must lie on the plane spanned by $\nabla g(P)$ and $\nabla h(P)$.

So, $\nabla f(P) = \lambda_1 \nabla g(P) + \lambda_2 \nabla h(P)$ for some constants λ_1 and λ_2.

Remark: We may generalize to case with more constraints.

Example 4:

The plane $x + y + z = 12$ interesects the paraboloid $z = x^2 + y^2$ in an ellipse.
Find the highest and lowest points on this ellipse.



Solutions

Let $f(x, y, z) = z$ for any $(x, y, z) \in R^3$.
Let $g(x, y, z) = x + y + z - 12$ for any $(x, y, z) \in R^3$.
Let $h(x, y, z) = z - x^2 - y^2$ for any $(x, y, z) \in R^3$.
We consider "Maximize $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$" AND "Minimize $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$".

$\nabla f(x, y, z) = (0,0,1)$; $\nabla g(x, y, z) = (1,1,1)$; $\nabla h(x, y, z) = (-2x, -2y, 1)$
$\nabla f(x, y, z) = \lambda_1 \nabla g(x, y, z) + \lambda_2 \nabla h(x, y, z)$
$(0,0,1) = \lambda_1(1,1,1) + \lambda_2(-2x, -2y, 1)$

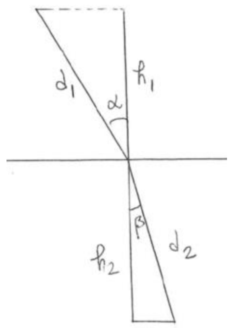
So, $\begin{cases} \lambda_1 - 2\lambda_2 x = 0 \\ \lambda_1 - 2\lambda_2 y = 0. \\ \lambda_1 + \lambda_2 = 1 \end{cases}$

From the first two equations, we have $x = \frac{\lambda_1}{2\lambda_2} = y$.
$g(x, y, z) = 0 \Rightarrow x + y + z - 12 = 0 \Rightarrow z = 12 - 2x$
$h(x, y, z) = 0 \Rightarrow z - x^2 - y^2 = 0 \Rightarrow z = 2x^2$
Put $2x^2 = 12 - 2x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0 \Rightarrow x = -3$ or 2.
The points are $(-3, -3, 18)$ and $(2,2,8)$.
The highest point is $(-3, -3, 18)$ and the lowest point is $(2,2,8)$.

<u>Applications</u>

<u>Example 1 (Shell's Law):</u>

A traveller (initially started at a fixed point h_1 units above a line L) has to go through the line to get to another fixed point h_2 units below the line L in minimum time. Suppose his speed is constantly v_1 above the line and constantly v_2 below the line. Show that the condition for the minimum time path is $\frac{v_1}{v_2} = \frac{sin\alpha}{sin\beta}$, where α is the angle of incidence and β is the angle of reflection.



<u>Proof</u>

$cos\alpha = \frac{h_1}{d_1}; d_1 = h_1 sec\alpha; cos\beta = \frac{h_2}{d_2}; d_2 = h_2 sec\beta$

Let $T(\alpha, \beta) = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{h_1}{v_1} sec\alpha + \frac{h_2}{v_2} sec\beta$.

Note that: $h_1 tan\alpha + h_2 tan\beta$ must be a constant (From a fixed point to another fixed point), say C.

Let $g(\alpha, \beta) = h_1 tan\alpha + h_2 tan\beta - C$.

We consider "Minimize $T(\alpha, \beta)$ subject to $g(\alpha, \beta) = 0$".

$\nabla T(\alpha, \beta) = \left(\frac{h_1}{v_1} tan\alpha \cdot sec\alpha, \frac{h_2}{v_2} tan\beta \cdot sec\beta\right); \nabla g(\alpha, \beta) = (h_1 sec^2\alpha, h_2 sec^2\beta)$

$$\nabla T(\alpha, \beta) = \lambda \nabla g(\alpha, \beta) \Longrightarrow \left(\frac{h_1}{v_1} tan\alpha \cdot sec\alpha, \frac{h_2}{v_2} tan\beta \cdot sec\beta\right) = \lambda(h_1 sec^2\alpha, h_2 sec^2\beta)$$

$$\begin{cases} \frac{h_1}{v_1} tan\alpha \cdot sec\alpha = \lambda h_1 sec^2\alpha \\ \frac{h_2}{v_2} tan\beta \cdot sec\beta = \lambda h_2 sec^2\beta \end{cases}$$

So, $\lambda = \frac{sin\alpha}{v_1} = \frac{sin\beta}{v_2}$.

Thus, $\frac{v_1}{v_2} = \frac{sin\alpha}{sin\beta}$

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

Example 2 (Arithmetic-Geometric Mean Inequality):

(i) Suppose that $x_1, x_2, \cdots, x_n$ are positive. Show that the minimum value of $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ subject to the constraint $x_1 \cdot x_2 \cdot \cdots \cdot x_n = 1$ is n.

(ii) Given n positive numbers $a_1, a_2, \cdots, a_n$, let $x_i = \frac{a_i}{(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{1/n}}$ for $i = 1,2,\cdots,n$ and apply the result in part (i) to deduce the arithmetic-geometric mean inequality:

$$\sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Proof of (ii)

As $x_i = \frac{a_i}{(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{1/n}}$ for $i = 1,2,\cdots,n$,

$$x_1 \cdot x_2 \cdot \cdots \cdot x_n = \prod_{i=1}^{n} \frac{a_i}{(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}}} = \frac{a_1 \cdot a_2 \cdot \cdots \cdot a_n}{a_1 \cdot a_2 \cdot \cdots \cdot a_n} = 1$$

$$x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} \frac{a_i}{(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}}} = \frac{a_1 + a_2 + \cdots + a_n}{(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}}}$$

By part(i), $\frac{a_1 + a_2 + \cdots + a_n}{(a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}}} \geq n$

Thus, $\sqrt[n]{a_1 \cdot a_2 \cdot \cdots \cdot a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$

Proof of (i)

Let $f: R^n \to R$ be defined by $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n$ and let $g: R^n \to R$ be defined by $g(x_1, x_2, \cdots, x_n) = x_1 \cdot x_2 \cdot \cdots \cdot x_n - 1$. Let $q_i = \frac{x_1 \cdot x_2 \cdot \cdots \cdot x_n}{x_i}$ for $i = 1,2,\cdots,n$.

We consider "Minimize $f(x_1, x_2, \cdots, x_n)$ subject to $g(x_1, x_2, \cdots, x_n) = 0$".

$\nabla f(x_1, x_2, \cdots, x_n) = (1,1,\cdots,1)$, $\nabla g(x_1, x_2, \cdots, x_n) = (q_1, q_2, \cdots, q_n)$

$\nabla f(x_1, x_2, \cdots, x_n) = \lambda \nabla g(x_1, x_2, \cdots, x_n)$

$\Longrightarrow (1,1,\cdots,1) = \lambda(q_1, q_2, \cdots, q_n) \Longrightarrow x_1 = x_2 = \cdots = x_n = \lambda \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$

Also $x_1 \cdot x_2 \cdot \cdots \cdot x_n - 1 = 0$, we have ${x_1}^n = 1$. Hence, $x_1 = 1$ (as $x_1 > 0$)

Thus, $x_1 = x_2 = \cdots = x_n = 1$ and $f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n = n$.

The minimum value is n.

Course Code: **MATH 2000**
Course Name: **Engineering Mathematics I**

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

 # Critical Points of Functions of One Variable
 # Second Derivative Test
Let f be a real-valued function on x and let $c, \delta \in R$ with $\delta > 0$.
Suppose f'' is continuous on $(c-\delta, c+\delta)$ AND $f'(c) = 0$.
We have:
(i) If $f''(c) > 0$, then $\left(c, f(c)\right)$ is **a** local minima.
(ii) If $f''(c) < 0$, then $\left(c, f(c)\right)$ is **a** local maxima.
(iii) If $f''(c) = 0$, then we have NO conclusions on the nature of $\left(c, f(c)\right)$.

 # Critical Points of Functions of Two Variables
Definition
Let $r \in R$ with $r > 0$ and $P(a, b) \in R^2$.
Suppose $f(x, y)$ is a continuously differentiable function defined on an open ball $B(P, r)$.
Note: $f_{xy}(P) = f_{yx}(P)$.
We say P is **a critical point** of f if $\nabla f(P) = (0,0)$.

Let $A = f_{xx}(P), B = f_{xy}(P) = f_{yx}(P), C = f_{yy}(P)$.
Let $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$.

Theorem (Two Variables Second Derivative Tests)
(i) If $A > 0$ and $\Delta > 0$, then $\left(a, b, f(a, b)\right)$ is a local minima
(ii) If $A < 0$ and $\Delta > 0$, then $\left(a, b, f(a, b)\right)$ is a local maxima
(iii) If $\Delta < 0$, then $\left(a, b, f(a, b)\right)$ is neither a local minima nor a local maxima.
It is called a saddle point.

Proof: Will be discussed later

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Course Name: **Engineering Mathematics I**

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

Example 1:

Locate and classify the critical points of $f(x,y) = 3x - x^3 - 3xy^2$.

Solution

As f is a polynomial in x and y, f is continuously differentiable on R^2.
$\nabla f(x,y) = (3 - 3x^2 - 3y^2, -6xy)$
$f_x(x,y) = 0 \Leftrightarrow 3 - 3x^2 - 3y^2 = 0 \Leftrightarrow x^2 + y^2 = 1$
$f_y(x,y) = 0 \Leftrightarrow -6xy = 0 \Leftrightarrow x = 0$ or $y = 0$
$\nabla f(x,y) = (0,0) \Leftrightarrow (x,y) = (0,1)$ or $(0,-1)$ or $(1,0)$ or $(-1,0)$
The critical points of f on R^2 are $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$.
$f_{xx}(x,y) = -6x$; $f_{xy}(x,y) = f_{yx}(x,y) = -6y$; $f_{yy}(x,y) = -6x$

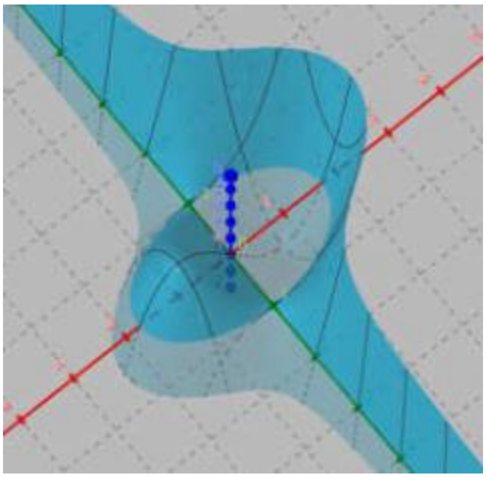
$$\Delta(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} -6x & -6y \\ -6y & -6x \end{vmatrix} = 36x^2 - 36y^2$$

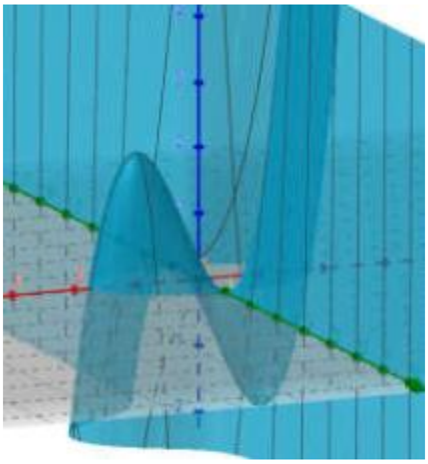
(i) Consider the critical point $(1,0)$
$f_{xx}(1,0) = -6 < 0$, $\Delta(1,0) = 36 > 0$, $(1,0,2)$ is a local maxima

(ii) Consider the critical point $(-1,0)$
$f_{xx}(-1,0) = 6 > 0$, $\Delta(-1,0) = 36 > 0$, $(-1,0,-2)$ is a local minima

(iii) Consider the critical point $(0,1)$
$f_{xx}(0,1) = 0$, $\Delta(0,1) = -36 < 0$, $(0,1,0)$ is a saddle point

(iv) Consider the critical point $(0,-1)$
$f_{xx}(0,-1) = 0$, $\Delta(0,-1) = -36 < 0$, $(0,-1,0)$ is a saddle point





Example 2:

Locate and classify the critical points of $f(x,y) = 6xy^2 - 2x^3 - 3y^4$.

Solution

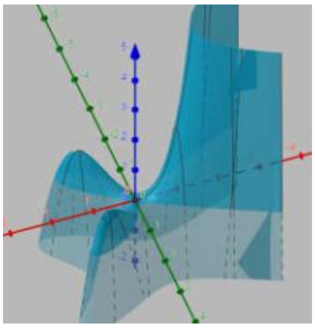
As f is a polynomial in x and y, f is continuously differentiable on R^2.
$\nabla f(x,y) = (6y^2 - 6x^2, 12xy - 12y^3)$
$f_x(x,y) = 0 \Leftrightarrow 6y^2 - 6x^2 = 0 \Leftrightarrow x^2 = y^2 \Leftrightarrow x = y$ or $x = -y$
$f_y(x,y) = 0 \Leftrightarrow 12xy - 12y^3 = 0 \Leftrightarrow 12y(x - y^2) = 0 \Leftrightarrow y = 0$ or $x = y^2$
For $x = y^2$ and $x = y$, we have $(x,y) = (0,0)$ or $(x,y) = (1,1)$
For $x = y^2$ and $x = -y$, we have $(x,y) = (0,0)$ or $(x,y) = (1,-1)$
$\nabla f(x,y) = (0,0) \Leftrightarrow (x,y) = (0,0)$ or $(1,1)$ or $(1,-1)$
The critical points of f on R^2 are $(0,0)$, $(1,1)$ and $(1,-1)$.
$f_{xx}(x,y) = -12x$; $f_{xy}(x,y) = f_{yx}(x,y) = 12y$; $f_{yy}(x,y) = 12x - 36y^2$

$$\Delta(x,y) = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} -12x & 12y \\ 12y & 12x - 36y^2 \end{vmatrix}$$

(i) Consider the critical point $(0,0,0)$
$f_{xx}(0,0) = 0$, $\Delta(0,0) = 0$
The test fails.
$f(0,0) = 0$
$f(0,y) = -3y^4 < 0$ when $y \neq 0$ amd $y \approx 0$
$f(x,0) = -2x^3 > 0$ when $x < 0$ and $x \approx 0$
$(0,0,0)$ is neither a local maxima nor a local minima. It is a saddle point.

(ii) Consider the critical point $(1,1)$
$f_{xx}(1,1) = -12 < 0$, $\Delta(1,1) = \begin{vmatrix} -12 & 12 \\ 12 & -24 \end{vmatrix} = 144 > 0$, $(1,1,1)$ is a local maxima

(iii) Consider the critical point $(1,-1)$
$f_{xx}(1,-1) = -12 < 0$, $\Delta(1,-1) = \begin{vmatrix} -12 & -12 \\ -12 & -24 \end{vmatrix} = 144 > 0$, $(1,-1,1)$ is a local maxima

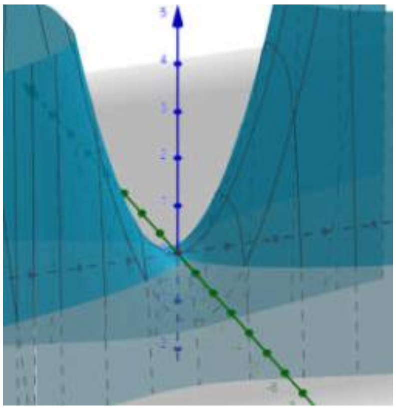


Example 3:

Locate and classify the critical points of $f(x, y) = x^2 - y^4$.

Solution

As f is a polynomial in x and y, f is continuously differentiable on R^2.
$\nabla f(x, y) = (2x, -4y^3)$
$\nabla f(x, y) = (0,0) \Leftrightarrow (2x, -4y^3) = (0,0) \Leftrightarrow (x, y) = (0,0)$
The critical point of f on R^2 is $(0,0)$
$f_{xx}(x, y) = 2$; $f_{xy}(x, y) = f_{yx}(x, y) = 0$; $f_{yy}(x, y) = -12y^2$
$\Delta(x, y) = \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -12y^2 \end{vmatrix}$
$f_{xx}(0,0) = 0$, $\Delta(0,0) = 0$
The test fails.
$f(0,0) = 0$
$f(0, y) = -y^4 < 0$ when $y \neq 0$ and $y \approx 0$
$f(x, 0) = x^2 > 0$ when $x \neq 0$ and $x \approx 0$
$(0,0,0)$ is neither a local maxima nor a local minima. It is a saddle point.



Example 4:

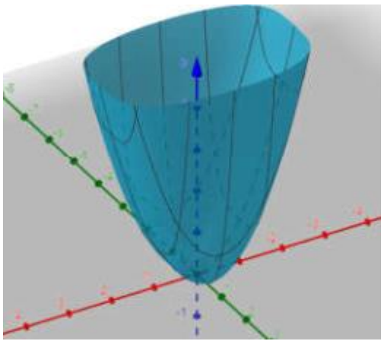
Locate and classify the critical points of $f(x, y) = x^2 + y^4$.

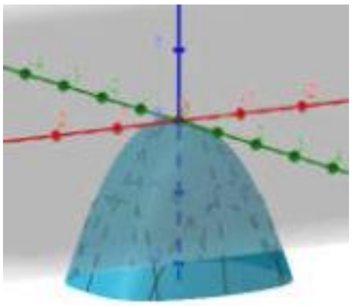
Solution

As f is a polynomial in x and y, f is continuously differentiable on R^2.
$\nabla f(x, y) = (2x, 4y^3)$
$\nabla f(x, y) = (0,0) \Leftrightarrow (2x, 4y^3) = (0,0) \Leftrightarrow (x, y) = (0,0)$
The critical point of f on R^2 is $(0,0)$
$f_{xx}(x, y) = 2$; $f_{xy}(x, y) = f_{yx}(x, y) = 0$; $f_{yy}(x, y) = 12y^2$

$$\Delta(x, y) = \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 12y^2 \end{vmatrix}$$

$f_{xx}(0,0) = 0$, $\Delta(0,0) = 0$
The test fails.
$f(0,0) = 0$
$f(x, y) = x^2 + y^4 \geq 0$ for any $(x, y) \in R^2$
$(0,0,0)$ is a local minima.





Exercise 5:

Locate and classify the critical points of $f(x, y) = -x^2 - y^4$.

Answer

The critical point of f on R^2 is $(0,0)$.
$(0,0,0)$ is a local maxima.

 # Behaviour of Quadratic Form

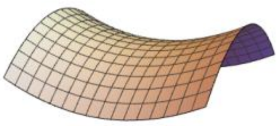
Let $Q(h,k) = Ah^2 + 2Bhk + Ck^2$.
Suppose $A \neq 0$. Let $\Delta = AC - B^2$.
Then, $Q(h,k) = \frac{1}{A}[(Ah + Bk)^2 + \Delta k^2]$.

Theorems:

(i) If $A > 0$ and $\Delta > 0$, then $(0,0,0)$ is a local minima.
Proof:
$Q(h,k) \geq 0 = Q(0,0)$

(ii) If $A < 0$ and $\Delta > 0$, then $(0,0,0)$ is a local maxima.
Proof:
$Q(h,k) \leq 0 = Q(0,0)$

(iii) If $\Delta < 0$, then $(0,0,0)$ is neither a local minima nor a local maxima.
Proof:
Case 1: $A > 0$
We can choose $k > 0$ and $k \approx 0$ so that $(Ah + Bk)^2 + \Delta k^2 > 0$ and $\|(h,k)\|$ is small.
We can choose $h > 0$ and $h \approx 0$ so that $(Ah + Bk)^2 + \Delta k^2 < 0$ and $\|(h,k)\|$ is small.
Case 2: $A < 0$
Omitted (As Exercise)



Course Code: **MATH 2000**
Course Name: **Engineering Mathematics I**

Lecture Notes for Chapter 12: Differentiation of Functions of Several Variables

 # Taylor's Formula for One Variable

Let $\phi \neq I \subset R$ and I is an open interval. Let $a, x \in I$.
Suppose f is a function defined on I.
Suppose $f, f', f'', f''', \cdots$ are continuous on I.
Then, $f(x) = f(a) + \left[\sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i\right] + R_{n+1}$
where $R_{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.
Roughly Speaking, $f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2$ when $x - a \approx 0$

 # Taylor's Formula for Two Variables

Let $f(t) = F(a+th, b+tk)$.
Then, $F(a+h, b+k) = f(1) = f(0) + \left[\sum_{i=1}^{n} \frac{f^{(i)}(0)}{i!}\right] + R_{n+1}$

$f(0) = F(a,b)$

$f'(t) = F_x(a+th, b+tk)\cdot h + F_y(a+th, b+tk)\cdot k$

$f'(0) = F_x(a,b)\cdot h + F_y(a,b)\cdot k$

$$f''(t) = \frac{d}{dt}\left[F_x(a+th, b+tk)\cdot h + F_y(a+th, b+tk)\cdot k\right]$$

$= F_{xx}(a+th, b+tk)\cdot h^2 + 2F_{xy}(a+th, b+tk)\cdot hk + F_{yy}(a+th, b+tk)\cdot k^2$

$f''(0) = F_{xx}(a,b)\cdot h^2 + 2F_{xy}(a,b)\cdot hk + F_{yy}(a,b)\cdot k^2$

We can show that

$$F(a+h, b+k) = F(a,b) + \left[\sum_{n=1}^{N}\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\cdot \frac{\partial^n F}{\partial x^{n-j}\partial y^j}\bigg|_{(x,y)=(a,b)} \cdot h^{n-j}k^j\right] + R_{N+1}$$

Roughly Speaking,

$F(a+h, b+k)$

$\approx F(a,b) + F_x(a,b)\cdot h + F_y(a,b)\cdot k + \frac{1}{2}\left[F_{xx}(a,b)\cdot h^2 + 2F_{xy}(a,b)\cdot hk + F_{yy}(a,b)\cdot k^2\right]$ when $\|(h,k)\| \approx 0$

Suppose $\nabla F(a,b) = (0,0)$.
Then, $F(a+h, b+k) \approx F(a,b) + \frac{1}{2}\left[F_{xx}(a,b)\cdot h^2 + 2F_{xy}(a,b)\cdot hk + F_{yy}(a,b)\cdot k^2\right]$ when $\|(h,k)\| \approx 0$

Let $A = F_{xx}(a,b)$, $B = F_{xy}(a,b) = F_{yx}(a,b)$ and $C = F_{yy}(a,b)$.

$F(a+h, b+k) - F(a,b) \approx \frac{1}{2}[Ah^2 + 2Bhk + Ck^2]$ when $\|(h,k)\| \approx 0$

It behaves like a quadratic form. Thus,

(i) If $A > 0$ and $\Delta > 0$, then $(a, b, F(a,b))$ is a local minima.
(ii) If $A < 0$ and $\Delta > 0$, then $(a, b, F(a,b))$ is a local maxima.
(iii) If $\Delta < 0$, then $(a, b, F(a,b))$ is neither a local minima nor a local maxima. It is called a saddle point.